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Generalized Poincaré algebras and Lovelock–Cartan gravity theory

P.K. Concha, D.M. Peñafiel, E.K. Rodríguez, P. Salgado *

Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile

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ABSTRACT

We show that the Lagrangian for Lovelock–Cartan gravity theory can be reformulated as an action which leads to General Relativity in a certain limit. In odd dimensions the Lagrangian leads to a Chern–Simons theory invariant under the generalized Poincaré algebra $\mathfrak{B}_{2n+1}$, while in even dimensions the Lagrangian leads to a Born–Infeld theory invariant under a subalgebra of the $\mathfrak{B}_{2n+1}$ algebra. It is also shown that torsion may occur explicitly in the Lagrangian leading to new torsional Lagrangians, which are related to the Chern–Pontryagin character for the B_{2n+1} group.

1. Introduction

The most general metric theory of gravity satisfying the criteria of general covariance and yielding to second-order field equations is a polynomial of degree $[d/2]$ in the curvature known as the Lanczos–Lovelock gravity theory (LL) [1,2]. The LL action can be written as the most general d-form invariant under local Lorentz transformations, constructed with the spin connection, the vielbein and their exterior derivatives, without the Hodge dual [3,4],

$$S=\int\sum_{p=0}^{[d/2]}\tilde{\alpha}_p\varepsilon_{a_1a_2\cdots a_d}R^{a_1a_2}\cdots R^{a_{2p-1}a_{2p}}e^{a_{2p+1}}\cdots e^{a_d}, \tag{1}$$

where $R^{ab}=d\omega^{ab}+\omega^a{}_c\omega^{cb}$ is the Lorentz curvature, e^a corresponds to the one-form vielbein and the coefficients $\tilde{\alpha}_p$, $p=0,1,\ldots,[d/2]$, are arbitrary constants and they are not fixed from first principles.

It is an accepted fact that requiring the LL theory to have the maximum possible number of degrees of freedom fixes the parameters $\tilde{\alpha}_p$'s in terms of the gravitational and the cosmological constants [5]. As a consequence, the action in odd dimensions can be formulated as a Chern–Simons (*ChS*) theory of the *AdS* group, while in even dimensions the action has a Born–Infeld (*BI*) form invariant only under local Lorentz rotations in the same way as the Einstein–Hilbert action [5–8].

Although the Einstein–Hilbert term is contained in the LL action, the *ChS* gravity for the *AdS* group and the *BI* gravity for the Lorentz group are dynamically very different from standard General Relativity.

In Ref. [9] it was shown that the standard, odd-dimensional General Relativity can be obtained from a Chern–Simons gravity theory for a certain $\mathfrak{B}_m$ Lie algebra, which will be called generalized Poincaré algebra[1] (where the particular case $\mathfrak{B}_4$ corresponds to the so-called Maxwell algebra [10]). The generalized Poincaré algebras can be obtained by a resonant reduced S-expansion of the *AdS* Lie algebra using $S_E^{(N)}=\{\lambda_\alpha\}_{\alpha=0}^{N+1}$ as semigroup [9].

The S-expansion method has been introduced in Ref. [13] (see also [14–16]) and consists in a powerful tool in order to obtain new Lie algebras from original ones. The method is based on combining the structure constants of a Lie algebra $\mathfrak{g}$ with the inner multiplication law of a semigroup S. The new Lie algebra $\mathfrak{G}=S\times\mathfrak{g}$ is called the S-expanded algebra. Interestingly, when a decomposition of the semigroup $S=\bigcup_{p\in I}S_p$ (where I is a set of indices) satisfies the same structure that the subspaces V_p of the original algebra $\mathfrak{g}=\bigoplus_{p\in I}V_p$, we say that $\mathfrak{G}_R=\bigoplus_{p\in I}S_p\times V_p$ is a resonant subalgebra of $\mathfrak{G}=S\times g$. In particular, when the semigroup has a zero element 0_S, the reduced algebra is obtained imposing $0_S\times\mathfrak{g}=0$.

Subsequently, in Ref. [11] it was found that standard even-dimensional General Relativity emerges as a limit of a Born–Infeld theory invariant under a certain subalgebra $\mathfrak{L}^{\mathfrak{B}_m}$ of the $\mathfrak{B}_m$ Lie algebra. These odd- and even-dimensional theories are described by the so-called Einstein–Chern–Simons (*EChS*) and the Einstein–Born–Infeld (*EBI*) actions, respectively.

* Corresponding author.
E-mail address: pasalgad@udec.cl (P. Salgado).

[1] Alternatively known as the Maxwell algebra type.

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Very recently it was found in Ref. [12] that standard odd- and even-dimensional General Relativity emerges as a weak coupling constant limit of a $(2p+1)$-dimensional Chern–Simons Lagrangian and of a $2p$-dimensional Born–Infeld Lagrangian invariant under $\mathfrak{B}_{2m+1}$ and $\mathfrak{L}^{\mathfrak{B}_{2m}}$, respectively, if and only if $m \geq p$.

It is the purpose of this paper to show that: (i) it is possible to reformulate the Lagrangian for Lovelock–Cartan gravity theory, which we call "Lagrangian of *Einstein–Lovelock–Cartan* (*ELC*)", such that, in odd dimensions leads to the Einstein–Chern–Simons Lagrangian, and in even dimensions leads to the Einstein–Born–Infeld Lagrangian; (ii) the torsion may occur explicitly in the Lagrangian and that, following a procedure analogous to that of Ref. [5], it is possible to find new torsional Lagrangians, which are related to the Chern–Pontryagin character for the $\mathfrak{B}_{2n+1}$ group.

This paper is organized as follows. In Section 2 we briefly review some aspects of the construction of the so-called generalized Poincaré algebras and how it is possible to obtain General Relativity from the Chern–Simons and Born–Infeld formalism using these algebras.

In Section 3 the *ELC*-Lagrangian is constructed. It is shown that this Lagrangian leads in odd dimensions to the *EChS* Lagrangian and in even dimensions leads to the *EBI* Lagrangian.

In Section 4 the *ELC*-Lagrangian is generalized adding torsion explicitly following a procedure analogous to that of Ref. [5]. It is shown that in $4p$ dimensions, the only $4p$-forms $\mathfrak{B}_{2n+1}$-invariant, constructed from $e^{(a,2k+1)}$, $R^{(ab,2k)}$ and $T^{(a,2k+1)}$ $(k=0,\dots,n-1)$, are Pontryagin type invariants $P_{(4p)}$.

In Section 5 we show that using the dual formulation of the S-expansion introduced in Ref. [17], it is possible to relate the Euler type invariant and the Pontryagin type invariant in $d=3$ dimensions. Section 6 concludes the work with a comment and possible developments.

2. General relativity and the generalized Poincaré algebras $\mathfrak{B}_{2n+1}$

In order to describe how the action for General Relativity can be obtained from the gravity actions invariant under generalized Poincaré algebras, let us review here the results obtained in Refs. [9,11,12]. Following the definitions of Ref. [13] let us consider the S-expansion of the Anti-de Sitter (*AdS*) Lie algebra using as a semigroup $S_E^{(2n-1)} = \{\lambda_0, \dots, \lambda_{2n}\}$ endowed with the multiplication law $\lambda_\alpha \lambda_\beta = \lambda_{\alpha+\beta}$ when $\alpha+\beta \leq 2n$; $\lambda_\alpha \lambda_\beta = \lambda_{2n}$ when $\alpha+\beta > 2n$. The $\tilde{J}_{ab}$, $\tilde{P}_a$ generators of the *AdS* algebra satisfy the following commutation relations

$$[\tilde{J}_{ab}, \tilde{J}_{cd}] = \eta_{bc}\tilde{J}_{ad} - \eta_{ac}\tilde{J}_{bd} - \eta_{bd}\tilde{J}_{ac} + \eta_{ad}\tilde{J}_{bc}, \tag{2}$$

$$[\tilde{J}_{ab}, \tilde{P}_c] = \eta_{bc}\tilde{P}_a - \eta_{ac}\tilde{P}_b, \tag{3}$$

$$[\tilde{P}_a, \tilde{P}_b] = \tilde{J}_{ab}, \tag{4}$$

where $a, b = 0, \dots, 2n$ and η_{ab} corresponds to the Minkowski metric. Let us consider the following subset decomposition $S_E^{(2n-1)} = S_0 \cup S_1$, with

$$S_0 = \{\lambda_{2m}, \text{ with } m = 0, \dots, n-1\} \cup \{\lambda_{2n}\}, \tag{5}$$

$$S_1 = \{\lambda_{2m+1}, \text{ with } m = 0, \dots, n-1\} \cup \{\lambda_{2n}\}, \tag{6}$$

where λ_{2n} corresponds to the zero element of the semigroup $(0_S = \lambda_{2n})$. After extracting a resonant subalgebra and performing its 0_S $(= \lambda_{2n})$-reduction, one finds the generalized Poincaré algebra $\mathfrak{B}_{2n+1}$,

$$[P_a, P_b] = Z_{ab}^{(1)}, \qquad [J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, \tag{7}$$

$$[J_{ab}, J_{cd}] = \eta_{cb}J_{ad} - \eta_{ca}J_{bd} + \eta_{db}J_{ca} - \eta_{da}J_{cb}, \tag{8}$$

$$\left[J_{ab}, Z_c^{(i)}\right] = \eta_{bc}Z_a^{(i)} - \eta_{ac}Z_b^{(i)}, \tag{9}$$

$$\left[Z_{ab}^{(i)}, P_c\right] = \eta_{bc}Z_a^{(i)} - \eta_{ac}Z_b^{(i)}, \tag{10}$$

$$\left[Z_{ab}^{(i)}, Z_c^{(j)}\right] = \eta_{bc}Z_a^{(i+j)} - \eta_{ac}Z_b^{(i+j)}, \tag{11}$$

$$\left[J_{ab}, Z_{cd}^{(i)}\right] = \eta_{cb}Z_{ad}^{(i)} - \eta_{ca}Z_{bd}^{(i)} + \eta_{db}Z_{ca}^{(i)} - \eta_{da}Z_{cb}^{(i)}, \tag{12}$$

$$\left[Z_{ab}^{(i)}, Z_{cd}^{(j)}\right] = \eta_{cb}Z_{ad}^{(i+j)} - \eta_{ca}Z_{bd}^{(i+j)} + \eta_{db}Z_{ca}^{(i+j)} - \eta_{da}Z_{cb}^{(i+j)}, \tag{13}$$

$$\left[P_a, Z_c^{(i)}\right] = Z_{ab}^{(i+1)}, \qquad \left[Z_a^{(i)}, Z_c^{(j)}\right] = Z_{ab}^{(i+j+1)}, \tag{14}$$

where $i, j = 1, \dots, n-1$. Let us note that the generators of the $\mathfrak{B}_{2n+1}$ algebra are related to the original ones through

$$J_{ab} = J_{(ab,0)} = \lambda_0 \otimes \tilde{J}_{ab}, \tag{15}$$

$$P_a = P_{(a,1)} = \lambda_1 \otimes \tilde{P}_a, \tag{16}$$

$$Z_{ab}^{(i)} = J_{(ab,2i)} = \lambda_{2i} \otimes \tilde{J}_{ab}, \tag{17}$$

$$Z_a^{(i)} = P_{(a,2i+1)} = \lambda_{2i+1} \otimes \tilde{P}_a, \tag{18}$$

then if $i > n-1$ we have $Z_{ab}^{(i)} = Z_a^{(i)} = 0$. The generalized Poincaré algebra $\mathfrak{B}_{2n+1}$ is also known as the Maxwell algebra type which was introduced in Ref. [12]. We note that setting $Z_{ab}^{(i+1)}$ and $Z_a^{(i)}$ equal to zero, we obtain the $\mathfrak{B}_4$ algebra which coincides with the Maxwell algebra $\mathcal{M}$ [10]. In fact, every generalized Poincaré algebra $\mathfrak{B}_l$ can be obtained from $\mathfrak{B}_{2n+1}$ setting some generators equal to zero. Besides, one can see that the commutators (8), (12) and (13) form a Lorentz type subalgebra of the $\mathfrak{B}_{2n+1}$ algebra. This subalgebra denoted as $\mathfrak{L}^{\mathfrak{B}_{2n+1}}$ can be obtained as an S-expansion of the Lorentz algebra $\mathfrak{L}$ using $S_0^{(2n-1)} = \{\lambda_0, \lambda_2, \lambda_4, \dots, \lambda_{2n}\}$ as the relevant semigroup [11].

The generalized Poincaré algebras are particularly interesting in the context of gravity since it was shown in [9] that standard odd-dimensional General Relativity may emerge as the weak coupling constant limit $(l \to 0)$ of a $(2n+1)$-dimensional Chern–Simons Lagrangian invariant under the $\mathfrak{B}_{2n+1}$ algebra,

$$L_{CS\,(2n+1)}^{\mathfrak{B}_{2n+1}} = \sum_{k=1}^{n} l^{2k-2} c_k \alpha_j \delta_{i_1+\cdots+i_{n+1}}^{j} \delta_{p_1+q_1}^{i_{k+1}} \cdots \delta_{p_{n-k}+q_{n-k}}^{i_n} \varepsilon_{a_1\cdots a_{2n+1}} \times R^{(a_1a_2,i_1)} \cdots R^{(a_{2k-1}a_{2k},i_k)} e^{(a_{2k+1},p_1)} e^{(a_{2k+2},q_1)} \cdots \times e^{(a_{2n-1},p_{n-k})} e^{(a_{2n},q_{n-k})} e^{(a_{2n+1},i_{n+1})}, \tag{19}$$

where

$$c_k = \frac{1}{2(n-k)+1} \binom{n}{k}$$

$$R^{(ab,2i)} = d\omega^{(ab,2i)} + \eta_{cd}\omega^{(ac,2j)}\omega^{(db,2k)}\delta_{j+k}^{i},$$

and α_j are arbitrary constants which appear as a consequence of the S-expansion process. Let us note that the S-expanded fields are related to the *AdS* fields $\{\tilde{e}^a, \tilde{\omega}^{ab}\}$ as follows,

$$e^{(a2j+1)} = \lambda_{2j+1} \otimes \tilde{e}^a,$$

$$\omega^{(ab,2j)} = \lambda_{2j} \otimes \tilde{\omega}^{ab},$$

where $j = 0, 1, \dots, n-1$. In a similar way, the S-expanded Lorentz curvature $R^{(ab,2i)}$ is related to the Lorentz curvature $\tilde{R}^{ab} = d\tilde{\omega}^{ab} + \tilde{\omega}^a{}_c\tilde{\omega}^{cb}$ as $R^{(ab,2i)} = \lambda_{2i}\tilde{R}^{ab}$.

Similarly, it was shown in [11] that standard even-dimensional General Relativity emerges as the weak coupling constant limit $(l \to 0)$ of a $(2n)$-dimensional Born–Infeld type Lagrangian invariant under a subalgebra[2] $\mathfrak{L}^{\mathfrak{B}_{2n}}$ of the $\mathfrak{B}_{2n+1}$ algebra,

[2] The Lorentz type algebra $\mathfrak{L}^{\mathfrak{B}_{2n}}$ is identical to $\mathfrak{L}^{\mathfrak{B}_{2n+1}}$.

$$L_{BI\ (2n)}^{\mathfrak{L}^{\mathfrak{B}_{2n}}} = \sum_{k=1}^{n} l^{2k-2} \frac{1}{2n} \binom{n}{k} \alpha_j \delta^{j}_{i_1+\cdots+i_n} \delta^{i_{k+1}}_{p_1+q_1} \cdots \delta^{i_n}_{p_{n-k}+q_{n-k}} \times \varepsilon_{a_1\cdots a_{2n}} R^{(a_1a_2,i_1)} \cdots R^{(a_{2k-1}a_{2k},i_k)} e^{(a_{2k+1},p_1)} \times e^{(a_{2k+2},q_1)} \cdots e^{(a_{2n-1},p_{n-k})} e^{(a_{2n},q_{n-k})}. \tag{20}$$

These results have recently been generalized in Ref. [12] in which the authors have shown that $L_{CS\ (2n+1)}^{\mathfrak{B}_{2m+1}}$ and $L_{BI\ (2n)}^{\mathfrak{L}^{\mathfrak{B}_{2m}}}$ lead to the Einstein–Hilbert Lagrangian in a weak coupling constant limit, if and only if $m \geq n$.

3. The Einstein–Lovelock–Cartan Lagrangian

We have seen that the S-expansion procedure allows the construction of Chern–Simons gravities in odd dimensions invariant under the $\mathfrak{B}_{2n+1}$ algebra and Born–Infeld type gravities in even dimensions invariant under the $\mathfrak{L}^{\mathfrak{B}_{2n+1}}$ algebra, leading to General Relativity in a certain limit. These gravities are called the Einstein–Chern–Simons theories [9] and the Einstein–Born–Infeld theories [11], respectively. These findings show that it could be possible to reformulate the Lagrangian for Lovelock–Cartan gravity theory such that, in a certain limit, it leads to the General Relativity theory.

In this section we show that it is possible to write a Lovelock–Cartan Lagrangian leading to the *EChS* Lagrangian in $d = 2n - 1$ invariant under the $\mathfrak{B}_{2n-1}$ algebra, and to the *EBI* Lagrangian in $d = 2n$ invariant under the $\mathcal{L}^{\mathfrak{B}_{2n}}$ algebra. For this purpose we shall use the useful properties of the S-expansion procedure using $S_E^{(d-2)}$ as the relevant semigroup.

The expanded action is given by

$$S_{\mathcal{ELC}} = \int \sum_{p=0}^{[d/2]} \mu_i \alpha_p L_{\mathcal{ELC}}^{(p,i)} \tag{21}$$

where α_p and μ_i, with $i = 0, \ldots, d-2$, are arbitrary constants and $L_{\mathcal{ELC}}^{(p,i)}$ is given by

$$L_{\mathcal{ELC}}^{(p,i)} = l^{d-2} \delta^{i}_{i_1+\cdots+i_{d-p}} \varepsilon_{a_1a_2\cdots a_d} R^{(a_1a_2,i_1)} \cdots \times R^{(a_{2p-1}a_{2p},i_p)} e^{(a_{2p+1},i_{p+1})} \cdots e^{(a_d,i_{d-p})}, \tag{22}$$

with

$$R^{(ab,2i)} = d\omega^{(ab,2i)} + \eta_{cd}\omega^{(ac,2j)}\omega^{(db,2k)}\delta^{i}_{j+k}. \tag{23}$$

The expanded fields $\{e^{(a,2i+1)}, \omega^{(ab,2i)}\}$ are related to the *AdS* fields $\{\tilde{e}^a, \tilde{\omega}^{ab}\}$ as follows

$$\omega^{(ab,2i)} = \lambda_{2i} \otimes \tilde{\omega}^{ab}, \tag{24}$$

$$e^{(a,2i+1)} = \lambda_{2i+1} \otimes \tilde{e}^a, \tag{25}$$

where $\lambda_\alpha \in S_E^{(d-2)}$, which is a semigroup that obey the following multiplication law (see Ref. [13]),

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \leq d-1, \\ \lambda_{d-1}, & \text{when } \alpha+\beta > d-1. \end{cases} \tag{26}$$

Following the same procedure of Ref. [5], we consider the variation of the action with respect to $e^{(a,i)}$ and $\omega^{(ab,i)}$. The variation of the action (21) leads to the following equations:

$$\varepsilon_a^{(i)} = \sum_{p=0}^{[(d-1)/2]} \mu_i \alpha_p (d-2p) \varepsilon_a^{(p,i)} = 0, \tag{27}$$

$$\varepsilon_{ab}^{(i)} = \sum_{p=1}^{[(d-1)/2]} \mu_i \alpha_p p (d-2p) \varepsilon_{ab}^{(p,i)} = 0, \tag{28}$$

where

$$\varepsilon_a^{(p,i)} := l^{d-2} \delta^{i}_{i_1+\cdots+i_{d-p-1}} \varepsilon_{ab_1\cdots b_{d-1}} R^{(b_1b_2,i_1)} \cdots R^{(b_{2p-1}b_{2p},i_p)} \times e^{(b_{2p+1},i_{p+1})} \cdots e^{(b_{d-1},i_{d-p-1})}, \tag{29}$$

$$\varepsilon_{ab}^{(p,i)} := l^{d-2} \delta^{i}_{i_1+\cdots+i_{d-p-1}} \varepsilon_{aba_3\cdots a_d} R^{(a_3a_4,i_1)} \cdots R^{(a_{2p-1}a_{2p},i_{p-1})} \times T^{(a_{2p+1},i_p)} e^{(a_{2p+2},i_{p+1})} \cdots e^{(a_d,i_{d-p-1})}, \tag{30}$$

and where $T^{(a,i)} = de^{(a,i)} + \eta_{dc}\omega^{(ad,j)}e^{(c,k)}\delta^{i}_{j+k}$ is the expanded 2-form torsion. Using the covariant exterior derivative $D = d + [A, \cdot]$ (where A corresponds to the one-form gauge connection $\mathfrak{B}_{2n-1}$-valued) and the Bianchi identity for the expanded 2-form curvature $DR^{(ab,i_j)} = 0$, we have

$$D\varepsilon_a^{(p,i)} = l^{d-2}(d-1-2p)\delta^{i}_{i_1+\cdots+i_{d-p-1}} \varepsilon_{ab_1\cdots b_{d-1}} \times R^{(b_1b_2,i_1)} \cdots R^{(b_{2p-1}b_{2p},i_p)} \times T^{(b_{2p+1},i_{p+1})} e^{(b_{2p+2},i_p)} \cdots e^{(a_{d-1},i_{d-p-1})}. \tag{31}$$

Since

$$e^{(b,j)} \varepsilon_{ba}{}^{(p,k)} \delta^{i}_{j+k} = l^{d-2} \delta^{i}_{i_1+\cdots+i_{d-p-1}} \varepsilon_{aa_1\cdots a_{d-2}} \times R^{(a_1a_2,i_1)} \cdots R^{(a_{2p-3}a_{2p-2},i_{p-1})} \times T^{(a_{2p-1},i_p)} e^{(a_{2p},i_{p+1})} \cdots e^{(a_{d-2},i_{d-p-1})}, \tag{32}$$

one finds

$$e^{(b,j)} \varepsilon_{ba}{}^{(p+1,k)} \delta^{i}_{j+k} = l^{d-2} \delta^{i}_{i_1+\cdots+i_{d-p}} \varepsilon_{aa_1\cdots a_{d-1}} \times R^{(a_1a_2,i_1)} \cdots R^{(a_{2p-1}a_{2p},i_{p-1})} \times T^{(a_{2p+1},i_p)} e^{(a_{2p+2},i_{p+1})} \cdots e^{(a_{d-1},i_{d-p})}. \tag{33}$$

From (31) and (33) we have

$$D\varepsilon_a^{(p,i)} = (d-1-2p) e^{(b,j)} \varepsilon_{ba}{}^{(p+1,k)} \delta^{i}_{j+k}$$

for $0 \leq p \leq [(d-1)/2]$. This means that

$$D\varepsilon_a{}^{(i)} = \sum_{p=0}^{[(d-1)/2]} \mu_i \alpha_p (d-2p)(d-1-2p) e^{(b,j)} \varepsilon_{ba}{}^{(p+1,k)} \delta^{i}_{j+k}. \tag{34}$$

if $p' = p+1$ we find

$$D\varepsilon_a{}^{(i)} = \sum_{p'=1}^{[(d+1)/2]} \mu_i \alpha_{p'-1} \big(d-2p'+2\big)\big(d-2p'+1\big) \times e^{(b,j)} \varepsilon_{ba}{}^{(p',k)} \delta^{i}_{j+k}, \tag{35}$$

which can be rewritten as

$$D\varepsilon_a{}^{(i)} = \sum_{p=1}^{[(d+1)/2]} \mu_i \alpha_{p-1} (d-2p+2)(d-2p+1) \times e^{(b,j)} \varepsilon_{ba}{}^{(p,k)} \delta^{i}_{j+k}, \tag{36}$$

which by consistency with $\varepsilon_a^{(i)} = 0$ must also vanish. Taking the product of $\varepsilon_{ba}{}^{(k)}$ with $e^{(b,j)}$ we find

$$e^{(b,j)} \varepsilon_{ba}{}^{(k)} \delta^{i}_{j+k} = \sum_{p=1}^{[(d-1)/2]} \mu_i \alpha_p p (d-2p) e^{(b,j)} \varepsilon_{ba}{}^{(p,k)} \delta^{i}_{j+k} \tag{37}$$

which vanishes by consistency with $\varepsilon_{ab}^{(i)} = 0$.

In general there are different ways of choosing the coefficients α_p which in general correspond to different theories with different numbers of degrees of freedom. It is possible to choose the α_p such that $\varepsilon_a{}^{(i)}$ and $\varepsilon_{ab}{}^{(i)}$ are independent. This last condition corresponds to the maximum number of independent components.

3.1. Chern–Simons gravity invariant under $\mathfrak{B}_{2n-1}$

Following the same procedure of Ref. [5] one can see that in the odd-dimensional case Eqs. (36), (37) lead to the coefficients given by

$$\alpha_p = \alpha_0 \frac{(2n-1)(2\gamma)^p}{(2n-2p-1)} \binom{n-1}{p}, \tag{38}$$

where α_0 and γ are related to the gravitational and the cosmological constants,

$$\alpha_0 = \frac{\kappa}{(l^{d-1}d)}; \qquad \gamma = -\operatorname{sgn}(\Lambda)\frac{l^2}{2}. \tag{39}$$

For any dimension d, l is a length parameter related to the cosmological constant by

$$\Lambda = \pm \frac{(d-1)(d-2)}{2l^2}, \tag{40}$$

and the gravitational constant G is related to κ through

$$\kappa^{-1} = 2(d-2)!\Omega_{d-2}G. \tag{41}$$

With these coefficients the Lagrangian (21) may be written as the Chern–Simons form

$$L^{\mathfrak{B}_{2n-1}}_{CS\,(2n-1)} = \sum_{p=0}^{n-1} l^{2p-2} \frac{\kappa}{2(n-p)-1} \binom{n-1}{p} \mu_i \delta^i_{i_1+\cdots+i_{2n-1-p}} \times \varepsilon_{a_1 a_2 \cdots a_{2n-1}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1} a_{2p}, i_p)} \times e^{(a_{2p+1}, i_{p+1})} \cdots e^{(a_{2n-1}, i_{2n-1-p})}. \tag{42}$$

Let us note that this Lagrangian can be expressed equivalently as follows[3]

$$L^{\mathfrak{B}_{2n-1}}_{CS\,(2n-1)} = \sum_{k=1}^{n-1} l^{2k-2} c_k \alpha_i \delta^i_{i_1+\cdots+i_n} \delta^{i_{k+1}}_{p_1+q_1} \cdots \delta^{i_{n-1}}_{p_{n-1-k}+q_{n-1-k}} \times \varepsilon_{a_1 \cdots a_{2n-1}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2k-1} a_{2k}, i_k)} e^{(a_{2k+1}, p_1)} \times e^{(a_{2k+2}, q_1)} \cdots e^{(a_{2n-3}, p_{n-1-k})} e^{(a_{2n-2}, q_{n-1-k})} e^{(a_{2n-1}, i_n)}, \tag{43}$$

where

$$c_k = \frac{1}{2(n-k)-1} \binom{n-1}{k}, \tag{44}$$

$$\alpha_i = \kappa \mu_i \tag{45}$$

and

$$R^{(ab,2i)} = d\omega^{(ab,2i)} + \eta_{cd}\omega^{(ac,2j)}\omega^{(db,2k)}\delta^i_{j+k}. \tag{46}$$

This is the Einstein–Chern–Simons Lagrangian [compare with Eq. (19)] found in Ref. [9].

[3] The term with $p=0$ does not contribute to the sum because $\delta^i_{i_1+\cdots+i_{2n-1}} = 0$ for any value of i and n.

3.2. Born–Infeld gravity invariant under $\mathcal{L}^{\mathfrak{B}_{2n}}$

In the even-dimensional case, following the same procedure of Ref. [5] one can see that Eqs. (36), (37), lead to the following coefficients

$$\alpha_p = \alpha_0 (2\gamma)^p \binom{n}{p}. \tag{47}$$

With these coefficients the Lagrangian (21) is given by

$$L^{\mathcal{L}^{\mathfrak{B}_{2n}}}_{BI\,(2n)} = \sum_{p=0}^{n} \frac{\kappa}{2n} l^{2p-2} \binom{n}{p} \mu_i \delta^i_{i_1+\cdots+i_{2n-p}} \times \varepsilon_{a_1 a_2 \cdots a_{2n}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2p-1} a_{2p}, i_p)} \times e^{(a_{2p+1}, i_{p+1})} \cdots e^{(a_{2n}, i_{2n-p})}, \tag{48}$$

or equivalently,[4]

$$L^{\mathcal{L}^{\mathfrak{B}_{2n}}}_{BI\,(2n)} = \sum_{k=1}^{n} \frac{1}{2n} l^{2k-2} \binom{n}{k} \alpha_i \delta^i_{i_1+\cdots+i_n} \delta^{i_{k+1}}_{p_1+q_1} \cdots \delta^{i_n}_{p_{n-k}+q_{n-k}} \times \varepsilon_{a_1 \cdots a_{2n}} R^{(a_1 a_2, i_1)} \cdots R^{(a_{2k-1} a_{2k}, i_k)} e^{(a_{2k+1}, p_1)} \times e^{(a_{2k+2}, q_1)} \cdots e^{(a_{2n-1}, p_{n-k})} e^{(a_{2n}, q_{n-k})}, \tag{49}$$

which corresponds to the Einstein–Born–Infeld Lagrangian found in Ref. [11]. It is important to note that the coefficients $\alpha_i = \kappa \mu_i$ are arbitrary constants.

In this way we have shown that the S-expansion procedure does not modify the α_p's coefficients defined in Ref. [5]. Unlike the Lanczos–Lovelock action, the expanded action (21) called the Einstein–Lovelock action, has the property of leading to General Relativity in a certain limit of the coupling constant l both even and odd dimensions.

4. Adding torsion in the Lagrangian

The Lagrangian (21) can be interpreted as the most general d-form invariant under a Lorentz type subalgebra $\mathfrak{L}^{\mathfrak{B}_{2n}}$ of the generalized Poincaré algebra. This Lagrangian is constructed from the expanded vielbein and the expanded spin connection $e^{(a,2k+1)}$, $\omega^{(ab,2k)}$ $(k=0,\ldots,n-1)$ and their exterior derivatives.[5]

One can see from the variation of the EL Lagrangian that Eq. (30) does not imply in $d>4$ the vanishing of the expanded torsion $T^{(a,2k+1)}$. The condition $T^{(a,2k+1)}=0$ implies that the expanded spin connection $\omega^{(ab,2k)}$ have a dependence on the expanded vielbein $e^{(a,2k+1)}$. Thus the expanded fields $\omega^{(ab,2k)}$ and $e^{(a,2k+1)}$ cannot be identified as the components of a connection for the generalized Poincaré algebra. Therefore, impose $T^{(a,2k+1)}=0$ seems to be restrictive and arbitrary. In this section, we study the possibility of adding terms which contain the expanded torsion to the ELC Lagrangian.

The Einstein–Lovelock–Cartan Lagrangian can be generalized adding torsion explicitly following a procedure analogous to that of the Refs. [5,18].

The only terms invariant under $\mathfrak{L}^{\mathfrak{B}_{2n}}$ that can be constructed out of $e^{(a,2k+1)}$, $\omega^{(ab,2k)}$ and their exterior derivatives, are $R^{(ab,2k)}$, $T^{(a,2k+1)}$, and products of them. Then the invariant combinations that can occur in the Lagrangian are:

[4] As in the odd-dimensional case $p=0$ does not contribute to the sum because $\delta^i_{i_1+\cdots+i_{2n}} = 0$ for any value of i and n.

[5] When $k=0$, $e^{(a,1)}$ and $\omega^{(ab,0)}$ are identified with the usual vielbein e^a and the spin connection ω^{ab}, respectively.

$$R_A^{(i)} = \delta^i_{2(k_1+\cdots+k_A)} R_{a_2}^{a_1,(2k_1)} \cdots R_{a_1}^{a_A,(2k_A)}, \tag{50}$$

$$V_A^{(i)} = \delta^i_{2(k_1+\cdots+k_A+k_{A+1}+k_{A+2}+1)} R_{a_2}^{a_1,(2k_1)} \cdots \times R_b^{a_A,(2k_A)} e_{a_1}{}^{,(2k_{A+1}+1)} e^{(b,2k_{A+2}+1)}, \tag{51}$$

$$T_A^{(i)} = \delta^i_{2(k_1+\cdots+k_A+k_{A+1}+k_{A+2}+1)} R_{a_2}^{a_1,(2k_1)} \cdots \times R_b^{a_A,(2k_A)} T_{a_1}{}^{,(2k_{A+1}+1)} T^{(b,2k_{A+2}+1)}, \tag{52}$$

$$K_A^{(i)} = \delta^i_{2(k_1+\cdots+k_A+k_{A+1}+k_{A+2}+1)} R_{a_2}^{a_1,(2k_1)} \cdots \times R_b^{a_A,(2k_A)} T_{a_1}{}^{,(2k_{A+1}+1)} e^{(b,2k_{A+2}+1)}, \tag{53}$$

where $i = 0, \ldots, 2n-2$. So that, the Lagrangian can be written as a linear combination of products of these basic invariant combinations. In a similar way to Ref. [18], we find that the Lagrangian has to be of the form

$$L = \sum_{p=0}^{[d/2]} \mu_i \alpha_p L_{\mathcal{EL}}^{(p,i)} + \sum_j \mu_i \beta_j L_{A_j}^{d,(i)}, \tag{54}$$

where the μ, α and β are constants, $L_{\mathcal{ELC}}^{(p,i)}$ corresponds to the Einstein–Lovelock Lagrangian (22) and $L_{A_j}^{d,(i)}$ is a d-form invariant under the $\mathfrak{L}^{\mathfrak{B}}$ algebra given by

$$L_{A_j}^{d,(i)} = R_{A_1}^{(i)} \cdots R_{A_r}^{(i)} T_{B_1}^{(i)} \cdots T_{B_t}^{(i)} V_{C_1}^{(i)} \cdots V_{C_v}^{(i)} K_{D_1}^{(i)} \cdots K_{D_k}^{(i)}. \tag{55}$$

Thus, the inclusion of the expanded torsion leads to a number of arbitrary coefficients βj. Interestingly, as in the *AdS* symmetry case, it is possible to choose the β's in order to enlarge the Lorentz type $\mathfrak{L}^{\mathfrak{B}}$ symmetry to the generalized Poincaré gauge symmetry.

In even dimensions, the $\mathfrak{B}_{2n+1}$-invariant d-forms are given by

$$\mathcal{P} = \langle F^{d/2} \rangle, \tag{56}$$

where $\langle \ldots \rangle$ denotes a symmetric invariant tensor for the $\mathfrak{B}_{2n+1}$ algebra. Here, $F = dA + AA$ is the 2-form curvature for the generalized Poincaré algebra and it is given by

$$F = \sum_{k=0}^{n-1} \left[\frac{1}{2} F^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} F^{(a,2k+1)} P_{(a,2k+1)} \right], \tag{57}$$

with

$$F^{(ab,2k)} = d\omega^{(ab,2k)} + \eta_{cd} \omega^{(ac,2i)} \omega^{(db,2j)} \delta^k_{i+j} + \frac{1}{l^2} e^{(a,2i+1)} e^{(b,2j+1)} \delta^k_{i+j+1}, \tag{58}$$

$$F^{(a,2k+1)} = de^{(a,2k+1)} + \eta_{bc} \omega^{(ab,2i)} e^{(c,2j)} \delta^k_{i+j}. \tag{59}$$

The $\omega^{(ab,2k)}$ and $e^{(a,2k+1)}$ are the different components of the 1-form connection A,

$$A = \sum_{k=0}^{n-1} \left[\frac{1}{2} \omega^{(ab,2k)} J_{(ab,2k)} + \frac{1}{l} e^{(a,2k+1)} P_{(a,2k+1)} \right], \tag{60}$$

where $J_{(ab,2k)}$ and $P_{(a,2k+1)}$ are the generators of the generalized Poincaré algebra $\mathfrak{B}_{2n+1}$.

Naturally, one of the invariants present in even dimensions is the Euler type invariant which is obtained from the following components of an invariant tensor,

$$\langle J_{(a_1 a_2, 2k_1)} \cdots J_{(a_{d-3} a_{d-2}, 2k_{(d-2)/2})} P_{(a_{d-1}, 2k_{d/2}+1)} \rangle = \mu_i \delta^i_{2(k_1+\cdots+k_{d/2})+1} \epsilon_{a_1 a_2 \cdots a_{d-1}}, \tag{61}$$

with $k_i = 0, \ldots, n-1$.

However, there are other components of the invariant tensor which lead to a different invariant known as the Pontryagin invariant which exists only in $4p$ dimensions. This invariant corresponds to the $\mathfrak{B}_{2n+1}$-invariant d-form built from $e^{(a,2k+1)}$, $R^{(ab,2k)}$, $T^{(a,2k+1)}$ and can be expressed as the exterior derivative of a Chern–Simons $(4p-1)$-form,

$$dL_{T\,(4p-1)}^{\mathfrak{B}_{2n+1}} = P_{(4p)}. \tag{62}$$

This implies that in odd dimensions there are two families of Lagrangians invariant under the generalized Poincaré algebra $\mathfrak{B}_{2n+1}$:

- The *Euler–Chern–Simons* form $L_{E\,(2p+1)}^{\mathfrak{B}_{2n+1}}$, in $D = 2p+1$. Its exterior derivative is the Euler density in $2p+2$ dimensions and does not involve torsion explicitly.
- The *Pontryagin–Chern–Simons* form $L_{T\,(4p-1)}^{\mathfrak{B}_{2n+1}}$, in $D = 4p-1$. Its exterior derivative is the Pontryagin invariant $P_{(4p)}^{\mathfrak{B}_{2n+1}}$ in $4p$ dimensions.

These results generalize those obtained in Ref. [5] to our case. The similitude is not a surprise since the $\mathfrak{B}_{2n+1}$ algebra corresponds to an expansion of the *AdS* algebra. Nevertheless, unlike the *AdS*-invariant gravity theory, the locally $\mathfrak{B}_{2n+1}$-invariant gravity theory leads to General Relativity in the weak coupling constant limit $(l \to 0)$ (see Refs. [9,11,12]).

Interestingly, in $4p$ dimensions, both families exist which allows us to write the most general Lagrangian for gravity in $d = 4p-1$ invariant under the generalized Poincaré algebra, namely

$$L_{CS\,(4p-1)}^{\mathfrak{B}_{2n+1}} = L_{E\,(4p-1)}^{\mathfrak{B}_{2n+1}} + L_{T\,(4p-1)}^{\mathfrak{B}_{2n+1}} \tag{63}$$

$$= \alpha_i L_{E\,(4p-1)}^{(i)} + \alpha_j L_{T\,(4p-1)}^{(j)}, \tag{64}$$

where $i = 1, 3, 5, \ldots, 2n-1$ and $j = 0, 2, 4, \ldots, 2n-2$. The α's are arbitrary and are a consequence of the S-expansion procedure. In the next subsection, we explore an example in $d = 3$ which clarifies this point.

4.1. Example for $d = 3$

Let us consider a $(2+1)$-dimensional Lagrangian invariant under the $\mathfrak{B}_5$ algebra. This algebra can be obtained from the *AdS* algebra, using the S-expansion procedure of Ref. [13].

After extracting a resonant subalgebra and performing a 0_S-reduction, one finds the $\mathfrak{B}_5$ algebra, whose generators satisfy the following commutation relations

$$[P_a, P_b] = Z_{ab}, \qquad [J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b \tag{65}$$

$$[J_{ab}, J_{cd}] = \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb} \tag{66}$$

$$[J_{ab}, Z_{cd}] = \eta_{cb} Z_{ad} - \eta_{ca} Z_{bd} + \eta_{db} Z_{ca} - \eta_{da} Z_{cb} \tag{67}$$

$$[J_{ab}, Z_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{68}$$

$$[Z_{ab}, P_c] = \eta_{bc} Z_a - \eta_{ac} Z_b, \tag{69}$$

$$[Z_{ab}, Z_c] = [Z_{ab}, Z_{cd}] = [P_a, Z_c] = 0. \tag{70}$$

In order to write down a Chern–Simons Lagrangian for the $\mathfrak{B}_5$ algebra, we start from the $\mathfrak{B}_5$-valued one-form gauge connection

$$A = \frac{1}{2} \omega^{ab} J_{ab} + \frac{1}{l} e^a P_a + \frac{1}{2} k^{ab} Z_{ab} + \frac{1}{l} h^a Z_a, \tag{71}$$

and the associated two-form curvature

$$F = \frac{1}{2} R^{ab} J_{ab} + \frac{1}{l} T^a P_a + \frac{1}{2}\left(D_\omega k^{ab} + \frac{1}{l^2} e^a e^b\right) Z_{ab} + \frac{1}{l}\left(D_\omega h^a + k^a{}_b e^b\right) Z_a. \quad (72)$$

Using Theorem VII.2 of Ref. [13], it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathfrak{B}_5$ algebra are given by

$$\langle J_{ab} J_{cd}\rangle_{\mathfrak{B}_5} = \alpha_0(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}),$$

$$\langle J_{ab} P_c\rangle_{\mathfrak{B}_5} = \alpha_1 \epsilon_{abc}, \quad (73)$$

$$\langle J_{ab} Z_{cd}\rangle_{\mathfrak{B}_5} = \alpha_2(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}),$$

$$\langle J_{ab} Z_c\rangle_{\mathfrak{B}_5} = \alpha_3 \epsilon_{abc}, \quad (74)$$

$$\langle P_a P_c\rangle_{\mathfrak{B}_5} = \alpha_2 \eta_{ac}, \qquad \langle Z_{ab} P_c\rangle_{\mathfrak{B}_5} = \alpha_3 \epsilon_{abc}, \quad (75)$$

where α_0, α_1, α_2 and α_3 are arbitrary constants.

Using these components of the invariant tensor in the general expression for the *ChS* Lagrangian $L_{ChS} = \langle AdA + \frac{2}{3}A^3\rangle$, we find that the *ChS* Lagrangian invariant under the $\mathfrak{B}_5$ algebra is given by

$$L^{\mathfrak{B}_5}_{ChS\,(2+1)} = \frac{1}{l}\varepsilon_{abc}\left[\alpha_1 R^{ab} e^c + \alpha_3\left(\frac{1}{3l^2} e^a e^b e^c + R^{ab} h^c + k^{ab} T^c\right)\right] + \frac{\alpha_0}{2}\left(\omega^a_{\ b} d\omega^b_{\ a} + \frac{2}{3}\omega^a{}_b \omega^b{}_c \omega^c{}_a\right) + \frac{\alpha_2}{2}\left(\frac{2}{l^2} e^a T_a + \omega^a_{\ b} dk^b_{\ a} + k^a_{\ b} d\omega^b_{\ a} + 2\omega^a{}_b \omega^b{}_c k^c{}_a\right) \quad (76)$$

$$= \alpha_1 L^{(1)}_{E\,(3)} + \alpha_3 L^{(3)}_{E\,(3)} + \alpha_0 L^{(0)}_{T\,(3)} + \alpha_2 L^{(2)}_{T\,(3)}. \quad (77)$$

The exterior derivative of this Lagrangian leads us to the following associated invariant

$$\mathcal{P}^{\mathfrak{B}_5}_{(4)} = \frac{1}{l}\varepsilon_{abc}\left[\alpha_1 R^{ab} T^c + \alpha_3\left(\frac{1}{l^2} e^a e^b T^c + R^{ab}\left(D_\omega h^c + k^c{}_d e^d\right) + D_\omega k^{ab} T^c\right)\right] + \frac{\alpha_0}{2} R^a{}_b R^b{}_a + \frac{\alpha_2}{2}\left[\frac{2}{l^2}\left(T^a T_a - e^a e^b R_{ab}\right) + 2R^a{}_b D_\omega k^b{}_a\right], \quad (78)$$

where in addition to an Euler type density we can see that appears the usual Pontryagin density $P_{(4)} = R^a{}_b R^b{}_a$, the Nieh–Yan $N_{(4)} = \frac{2}{l^2}(T^a T_a - e^a e^b R_{ab})$ and a Pontryagin type density $P_4(k) = 2R^a{}_b D_\omega k^b{}_a$ coming from the new fields.

Note that these densities $P_{(4)}$, $N_{(4)}$ and $P_{(4)}(k)$ are combined in a Pontryagin type invariant for the B_5 group which is written as follows (choosing $\alpha_0 = \alpha_2$)

$$F^A{}_B F^B{}_A = R^a{}_b R^b{}_a + \left[\frac{2}{l^2}\left(T^a T_a - e^a e^b R_{ab}\right) + 2R^a{}_b D_\omega k^b{}_a\right], \quad (79)$$

where

$$F^{AB} = \begin{pmatrix} R^{ab} + (D_\omega k^{ab} + \frac{1}{l^2} e^a e^b) & \frac{1}{l} T^a + \frac{1}{l}(D_\omega h^a + k^a{}_c e^c) \\ -\frac{1}{l} T^b - \frac{1}{l}(D_\omega h^b + k^b{}_c e^c) & 0 \end{pmatrix} \quad (80)$$

In the next section we show that the $\mathfrak{B}_5$-invariant Lagrangian (76) can be obtained directly from the Lorentz-invariant Lagrangian.

5. Relation between the Pontryagin and Euler invariants

In this section we show that it is possible to relate the Lorentz invariant Lagrangian which depends only on the spin connection, and the Lagrangian obtained for the $\mathfrak{B}_5$ algebra. This means that by dual formulation of the S-expansion [17] is possible to obtain both an Euler type invariant and a Pontryagin type invariant from the Pontryagin invariant.

Consider first the Lorentz algebra $\mathcal{L}$ in $(2+1)$-dimensions,

$$[J_{ab}, J_{cd}] = \eta_{cb} J_{ad} - \eta_{ca} J_{bd} + \eta_{db} J_{ca} - \eta_{da} J_{cb}. \quad (81)$$

The one-form gauge connection A and the associated two-form curvature F are given by

$$A = \frac{1}{2}\omega^{ab} J_{ab}, \quad (82)$$

$$F = \frac{1}{2} R^{ab} J_{ab}, \quad (83)$$

where $R^{ab} = d\omega^{ab} + \omega^a{}_c \omega^{cb}$ is the Lorentz curvature. The corresponding Chern–Simons Lagrangian invariant under the Lorentz algebra $\mathcal{L}$ is given by

$$L^{Lorentz}_3 = \omega^a{}_b d\omega^b{}_a + \frac{2}{3}\omega^a{}_b \omega^b{}_c \omega^c{}_a, \quad (84)$$

which can be written as

$$L^{Lorentz}_3 = \left\langle AdA + \frac{2}{3}A^3\right\rangle,$$

where the invariant tensor $\langle\cdots\rangle$ for the Lorentz algebra is

$$\langle J_{ab} J_{cd}\rangle_{\mathcal{L}} = \eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}.$$

Before starting the S-expansion of the Lorentz algebra is useful to define

$$J^a = -\frac{1}{2}\epsilon^{abc} J_{bc}, \qquad \omega_a = -\frac{1}{2}\epsilon_{abc}\omega^{bc},$$

so that

$$A = \omega_a J^a, \qquad F = F_a J^a, \quad (85)$$

with

$$F_a = -\frac{1}{2}\epsilon_{abc} R^{bc} = d\omega_a - \frac{1}{2}\eta_{ab}\epsilon^{bcd}\omega_c\omega_d. \quad (86)$$

Now let us consider the $S^{(3)}_E$ expansion of Lorentz algebra. The appropriate semigroup $S^{(3)}_E = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ is endowed with the following product:

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{when } \alpha+\beta \le 4, \\ \lambda_4, & \text{when } \alpha+\beta > 4, \end{cases} \quad (87)$$

where $\lambda_4 = 0_s$ is the zero element of the semigroup.

In a similar way to Ref. [17], we define the spin connection and the 2-form curvature as

$$\omega_a = \lambda_0 \omega_a{}^{(0)} + \lambda_1 \omega_a{}^{(1)} + \lambda_2 \omega_a{}^{(2)} + \lambda_3 \omega_a{}^{(3)}, \quad (88)$$

$$F_a = \lambda_0 F_a{}^{(0)} + \lambda_1 F_a{}^{(1)} + \lambda_2 F_a{}^{(2)} + \lambda_3 F_a{}^{(3)}, \quad (89)$$

where

$$\omega_a{}^{(0)} = \omega_a, \qquad \omega_a{}^{(2)} = k_a,$$

$$\omega_a{}^{(1)} = \frac{1}{l} e_a, \qquad \omega_a{}^{(3)} = \frac{1}{l} h_a,$$

and

$$F_a{}^{(0)} = -\frac{1}{2}\epsilon_{abc}R^{bc}, \tag{90}$$

$$F_a{}^{(1)} = \frac{1}{l}T_a, \tag{91}$$

$$F_a{}^{(2)} = -\frac{1}{2}\epsilon_{abc}\left(D_\omega k^{bc} + \frac{1}{l^2}e^b e^c\right), \tag{92}$$

$$F_a{}^{(3)} = \frac{1}{l}\left(D_\omega h^a + k^a{}_b e^b\right). \tag{93}$$

Here we identify e^a with the vielbein, R^{ab} with the Lorentz curvature, T^a with the torsion, and k^{ab} and h^a are identified as bosonic "matter" fields.

Using Theorem VII.2 of Ref. [13], it is possible to show that the only non-vanishing components of an invariant tensor for the $\mathfrak{B}_5$ algebra are given by

$$\langle J_{ab} J_{cd}\rangle = \alpha_0(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}), \tag{94}$$

$$\langle J_{ab} Z_{cd}\rangle = \alpha_2(\eta_{ad}\eta_{bc} - \eta_{ac}\eta_{bd}), \tag{95}$$

$$\langle P_a P_c\rangle = \alpha_2\eta_{ac}, \tag{96}$$

$$\langle J_{ab} P_c\rangle = \alpha_1\epsilon_{abc}, \tag{97}$$

$$\langle J_{ab} Z_c\rangle = \alpha_3\epsilon_{abc}, \tag{98}$$

$$\langle Z_{ab} P_c\rangle = \alpha_3\epsilon_{abc}, \tag{99}$$

where α_0, α_1, α_2 and α_3 are arbitrary constants. Now if we use the components of the invariant tensor (94)–(99) in the general expression for a Chern–Simons Lagrangian we find the $\mathfrak{B}_5$-invariant CS Lagrangian in $(2+1)$ dimensions,

$$\begin{aligned} L^{\mathfrak{B}_5}_{CS\,(2+1)} = {} & \frac{1}{l}\varepsilon_{abc}\left[\alpha_1 R^{ab}e^c + \alpha_3\left(\frac{1}{3l^2}e^a e^b e^c + R^{ab}h^c + k^{ab}T^c\right)\right] \\ & + \frac{\alpha_0}{2}\left(\omega^a_b d\omega^b_a + \frac{2}{3}\omega^a{}_b\omega^b{}_c\omega^c{}_a\right) \\ & + \frac{\alpha_2}{2}\left(\frac{2}{l^2}e^a T_a + \omega^a_b dk^b_a + k^a_b d\omega^b_a + 2\omega^a{}_b\omega^b{}_c k^c{}_a\right). \end{aligned} \tag{100}$$

The exterior derivative of this Lagrangian leads us to the following invariant polynomial,

$$\begin{aligned} P^{\mathfrak{B}_5}_{(4)} = {} & \frac{1}{l}\epsilon_{abc}\Big[\alpha_1 R^{ab}T^c \\ & + \alpha_3\left(\frac{1}{l^2}e^a e^b T^c + R^{ab}\left(D_\omega h^c + k^c{}_d e^d\right) + D_\omega k^{ab}T^c\right)\Big] \\ & + \frac{\alpha_0}{2}R^a{}_b R^b{}_a + \frac{\alpha_2}{2}\left[\frac{2}{l^2}\left(T^a T_a - e^a e^b R_{ab}\right) + 2R^a{}_b D_\omega k^b{}_a\right]. \end{aligned} \tag{101}$$

Thus we have shown that the S-expansion method allows us to relate the Pontryagin invariant of the Lorentz algebra with the invariants of the $\mathfrak{B}_5$ algebra studied in the previous section.

It is important to note that it is possible to generalize the previous result to the case of the $\mathfrak{B}_{2n+1}$ algebras. In fact, by considering the reduced $S_E^{(2n-1)}$-expansion of the Lorentz algebra $\mathcal{L}$ and using the Theorem VII.2 of Ref. [13] we can find the non-vanishing components of an invariant tensor for the expanded algebra and thus build a $(2+1)$-dimensional Lagrangian invariant under $\mathfrak{B}_{2n+1}$.

6. Comment and possible developments

In the present work we have shown that it is possible to construct an Einstein–Lovelock–Cartan Lagrangian that, in odd dimensions leads to the Einstein–Chern–Simons Lagrangian, and in even dimensions leads to the Einstein–Born–Infeld Lagrangian. The *EChS* and *EBI* theories are particularly interesting since it was shown in Refs. [9,11,12] that General Relativity can be obtained as a certain limit of these gravity theories. On the other hand we have shown that the Einstein–Lovelock–Cartan Lagrangian can be generalized adding torsional terms following a procedure analogous to that of Ref. [5]. Interestingly, the torsional terms appear explicitly in the Lagrangian only in $4p-1$ dimensions. Thus, the only $4p$-forms invariant under the generalized Poincaré algebra $\mathfrak{B}_{2n+1}$, constructed from $e^{(a,2k+1)}$, $R^{(ab,2k)}$ and $T^{(a,2k+1)}$ $(k = 0,\dots,n-1)$, are the Pontryagin invariants $P_{(4p)}$. Finally we have established a relation between the Pontryagin and the Euler invariants using the dual formulation of the S-expansion method introduced in Ref. [17].

The procedure considered here could play an important role in the context of supergravity in higher dimensions. In fact, it seems likely that it is possible to recover the standard odd- and even-dimensional supergravity from a Chern–Simons and Born–Infeld gravity theories, in a way very similar to the one shown here. In this way, the procedure sketched here could provide us with valuable information of what the underlyng geometric structure of supergravity could be (work in progress).

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References

[1] C. Lanczos, Ann. Math. 39 (1938) 842.
[2] D. Lovelock, J. Math. Phys. 12 (1971) 498.
[3] B. Zumino, Phys. Rep. 137 (1986) 109.
[4] C. Teitelboim, J. Zanelli, Dimensionally continued topological gravitation theory in Hamiltonian form, Class. Quantum Gravity 4 (1987) L125.
[5] R. Troncoso, J. Zanelli, Higher-dimensional gravity, propagating torsion and AdS gauge invariance, Class. Quantum Gravity 17 (2000) 4451.
[6] M. Bañados, C. Teitelboim, J. Zanelli, Lovelock–Born–Infeld theory of gravity, in: H. Falomir, R.E. Gamboa, P. Leal, F. Schaposnik (Eds.), J.J. Giambiagi Festschrift, La Plata, May 1990, World Scientific, Singapore, 1991.
[7] M. Bañados, C. Teitelboim, J. Zanelli, Dimensionally continued black holes, Phys. Rev. D 49 (1994) 975–986.
[8] J. Zanelli, Lecture notes on Chern–Simons (super-)gravities, second edition, arXiv:hep-th/0502193.
[9] F. Izaurieta, P. Minning, A. Perez, E. Rodriguez, P. Salgado, Standard general relativity from Chern–Simons gravity, Phys. Lett. B 678 (2009) 213.
[10] P. Salgado, S. Salgado, $\mathfrak{so}(D-1,1)\otimes\mathfrak{so}(D-1,2)$ algebras and gravity, Phys. Lett. B 728 (2014) 5.
[11] P.K. Concha, D.M. Peñafiel, E.K. Rodriguez, P. Salgado, Even-dimensional general relativity from Born–Infeld gravity, Phys. Lett. B 725 (2013) 419.
[12] P.K. Concha, D.M. Peñafiel, E.K. Rodriguez, P. Salgado, Chern–Simons and Born–Infeld gravity theories and Maxwell algebras type, Eur. Phys. J. C 74 (2014) 2741.
[13] F. Izaurieta, E. Rodriguez, P. Salgado, Expanding Lie (super)algebras through Abelian semigroups, J. Math. Phys. 47 (2006) 123512.
[14] M. Hatsuda, M. Sakaguchi, Prog. Theor. Phys. 109 (2003) 853, arXiv:hep-th/0106114.
[15] J.A. de Azcarraga, J.M. Izquierdo, M. Picón, O. Varela, Nucl. Phys. B 662 (2003) 185, arXiv:hep-th/0212347.
[16] J.A. de Azcarraga, J.M. Izquierdo, M. Picón, O. Varela, Class. Quantum Gravity 21 (2004) 1375, arXiv:hep-th/0401033.
[17] F. Izaurieta, A. Perez, E. Rodriguez, P. Salgado, Dual formulation of the Lie algebra S-expansion procedure, J. Math. Phys. 50 (2009) 073511.
[18] A. Mardones, J. Zanelli, Lovelock–Cartan theory of gravity, Class. Quantum Gravity 8 (1991) 1545.