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Generalized Maxwellian exotic Bargmann gravity theory in three spacetime dimensions

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ABSTRACT

We present a generalization of the so-called Maxwellian extended Bargmann algebra by considering a non-relativistic limit to a generalized Maxwell algebra defined in three spacetime dimensions. The non-relativistic Chern-Simons gravity theory based on this new algebra is also constructed and discussed. We point out that the extended Bargmann and its Maxwellian generalization are particular sub-cases of the generalized Maxwellian extended Bargmann gravity introduced here. The extension of our results using the semigroup expansion method is also discussed.

Contents

1. Introduction

There has been a growing interest in exploring non-relativistic (NR) gravity theories [1–28]. In three spacetime dimensions, gravity models can be formulated using the Chern-Simons (CS) formalism [29–31] offering a simpler framework to construct non-relativistic gravity actions. Furthermore, three-dimensional CS gravity theories can be seen as interesting toy models to approach higher-dimensional theories.

The construction of a proper finite NR CS action without degeneracy may require to enlarge the field content of the relativistic theory [32–34]. In the case of three-dimensional Einstein gravity without cosmological constant, it is necessary to consider two additional $U(1)$ gauge fields in order to define a consistent NR limit leading to the so-called extended Bargmann gravity [35,36]. The incorporation of a cosmological constant modifies the theory to the so-called extended Newton-Hooke gravity [37–43]. More recently, a NR version of a three-dimensional gravity theory coupled to electromagnetism has been presented in [44] describing what they called as Maxwellian extended Bargmann (MEB) gravity. Such NR theory requires to introduce three extra $U(1)$ gauge fields to the Maxwell algebra.

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The Maxwell algebra has been introduced in [45–47] in order to describe a Minkowski space in the presence of a electromagnetic field background. In three spacetime dimensions, a CS gravity action without cosmological constant invariant under the Maxwell algebra has been presented in [48–50] whose general solution and asymptotic structure have been studied in [51]. More recently, an isomorphic (dual) version of the Maxwell algebra denoted as Hietarinta-Maxwell algebra has been of particular interest for exploring spontaneous breaking of symmetry [52,53].

A generalization of the Maxwell algebra has been introduced in [54,55] and has been denoted as $\mathfrak{B}_5$ algebra. This generalization is characterized by the presence of an additional generator with respect to the Maxwell algebra. Interestingly, the aforesaid algebra belongs to a larger family of algebras denoted as $\mathfrak{B}_k$ where $\mathfrak{B}_4$ and $\mathfrak{B}_3$ are the Maxwell and Poincaré algebras, respectively. Such family has been useful to recover standard General Relativity without cosmological constant from a CS and Born-Infeld gravity theory [55–58]. Subsequently, the coupling of spin-3 gauge field to $\mathfrak{B}_k$ CS gravity models in three spacetime dimensions has been explored in [59].

It is natural to address the question whether such generalized Maxwell algebra admits a well-defined NR version in three spacetime dimensions. Here we show that the relativistic theory has to be enlarged with four $U(1)$ gauge fields in order to apply an Inönü-Wigner (IW) contraction [60,61] leading to a non-degenerate and finite NR CS gravity theory. The new symmetry obtained corresponds to a generalization of the MEB algebra and has been called GMEB algebra.

An alternative way to find the GMEB symmetry is also discussed considering the semigroup expansion method (S-expansion) [62–66] and following the procedure used in [67]. As was shown in [67], a generalized family of NR algebras, that we have denoted as generalized extended Bargmann algebra, can be obtained using the S-expansion procedure. In particular, we show that the extended Bargmann, the MEB and the GMEB algebras are particular sub-cases of this family of NR algebras. The expansion procedure considered here can be seen as a general method allowing to classify diverse NR symmetries by providing the proper NR limit and the additional gauge fields required in the relativistic theory. Interestingly, the S-expansion method provides not only with the commutation relations of the new NR algebras but also with the non-vanishing components of the invariant tensors which are essential to the construction of NR CS actions.

The paper is organized as follows: in Section 2, we give a brief review of the generalized Maxwell algebra. The corresponding relativistic CS action and its $U(1)$ enlargement are also presented. Sections 3 and 4 contain our main results. In particular, in Section 3 we present the contraction process leading to the GMEB gravity theory. The family of NR algebras obtained through the semigroup expansion procedure is presented in section 4. Section 5 concludes our work with some discussion about possible future developments.

2. Relativistic gravity and generalized Maxwell algebra

In this section we briefly review the generalized Maxwell algebra and present the construction of a three-dimensional CS gravity action invariant under such algebra.

A generalization of the Maxwell algebra has been first introduced as the $\mathfrak{B}_5$ algebra in [54,55]. It is characterized by the presence of the spacetime rotations J_A, the spacetime translations P_A, the so-called Maxwell gravitational generator Z_A and a new type of generator that we have denoted as N_A. The generators of the generalized Maxwell algebra satisfy the following non-vanishing commutation relations:

$$\begin{aligned} [J_A, J_B] &= \epsilon_{ABC} J^C, \qquad [P_A, P_B] = \epsilon_{ABC} Z^C, \\ [J_A, P_B] &= \epsilon_{ABC} P^C, \qquad [J_A, N_B] = \epsilon_{ABC} N^C, \\ [J_A, Z_B] &= \epsilon_{ABC} Z^C, \qquad [Z_A, P_B] = \epsilon_{ABC} N^C, \end{aligned} \tag{2.1}$$

where $A, B, C = 0, 1, 2$ are the Lorentz indices which are raised and lowered with the Minkowski metric. Here ϵ_{ABC} corresponds to the Levi Civita tensor which satisfies $\epsilon_{012} = -\epsilon^{012} = 1$. It is interesting to point out that the commutator $[P_a, P_b]$ is proportional to the Maxwell gravitational generator Z_A as in the Maxwell algebra. Nevertheless, the commutator $[Z_A, P_B]$ is no longer zero due to the presence of the new generator N_A. Furthermore, unlike the AdS-Lorentz algebra [68–70], this generalization is not a deformation of the Maxwell algebra and then does not reproduce the Maxwell symmetry through a contraction process.

Although this algebra and its generalizations have been explored with diverse applications, a three-dimensional CS gravity action based on this generalization of the Maxwell algebra has not been explicitly presented. A CS action in three spacetime dimensions reads

$$I[A] = \int \langle AdA + \frac{2}{3} A^3 \rangle, \tag{2.2}$$

where $\langle \cdots \rangle$ denotes the invariant trace and $A = A^a T_a$ corresponds to the gauge connection one-form. In our case, the connection one-form A is given by

$$A = W^A J_A + E^A P_A + K^A Z_A + U^A N_A, \tag{2.3}$$

where W^A is the spin connection one-form, E^A is the vielbein, K^A is the so-called gravitational Maxwell gauge field and U^A is the new gauge field along the Abelian generator N_A. The respective curvature two form $F = dA + \frac{1}{2}[A, A]$ reads

$$F = R^A(W) J_A + R^A(E) P_A + R^A(K) Z_A + R^A(U) N_A, \tag{2.4}$$

where the Lorentz curvature $R^A(W)$, the torsion $R^A(E)$ and the curvatures along the generators Z_A and N_A are respectively given by

$$\begin{aligned} R^A(W) &:= dW^A - \frac{1}{2}\epsilon^{ABC} W_B W_C, \\ R^A(E) &:= D_W E^A, \\ R^A(K) &:= D_W K^A - \frac{1}{2}\epsilon^{ABC} E_B E_C, \\ R^A(U) &:= D_W U^A - \epsilon^{ABC} K_B E_C. \end{aligned} \tag{2.5}$$

Here $D_W \Theta^A := d\Theta^A - \epsilon^{ABC} W_B \Theta_C$ is the usual Lorentz covariant derivative.

In order to construct the relativistic CS gravity action invariant under the algebra (2.1) we shall consider the most general non-vanishing components of the invariant tensor [55]

$$\begin{aligned} \langle J_A J_B \rangle &= \alpha_0 \eta_{AB}, \qquad \langle P_A P_B \rangle = \alpha_2 \eta_{AB}, \\ \langle J_A P_B \rangle &= \alpha_1 \eta_{AB}, \qquad \langle J_A N_B \rangle = \alpha_3 \eta_{AB}, \\ \langle J_A Z_B \rangle &= \alpha_2 \eta_{AB}, \qquad \langle Z_A P_B \rangle = \alpha_3 \eta_{AB}, \end{aligned} \tag{2.6}$$

where α_0, α_1, α_2 and α_3 are arbitrary constants. Then, considering the gauge connection one-form (2.3) and the invariant tensor (2.6) in the general form of a CS action (2.2), one gets

$$\begin{aligned} I_R = \int \Big[&\alpha_0 \left(W^A dW_A + \frac{1}{3}\epsilon_{ABC} W^A W^B W^C \right) + 2\alpha_1 E^A R_A(W) \\ &+ \alpha_2 \left(2K^A R_A(W) + E^A R_A(E) \right) \end{aligned}$$

$$+\alpha_3\left(2U^A R_A(W)+2E^A D_W K^A+\frac{1}{3}\epsilon^{ABC}E_A E_B E_C\right)\Big]. \tag{2.7}$$

One can see that such relativistic CS action contains four independent sectors proportional to α_0, α_1, α_2 and α_4. In particular, the term proportional to α_0 corresponds to the so-called exotic Einstein action [30]. The term along α_1 is the usual Einstein-Hilbert term and corresponds to the CS action based on the Poincaré symmetry. On the other hand, the Maxwellian gravitational gauge field has contribution in the Maxwell CS action proportional to α_2 [44,48–51,71,72] and to the new term α_3. The new gauge field U^A appears only along the α_3 term together to the cosmological constant term. Let us note that each sector is invariant under the generalized Maxwell algebra (2.1). Indeed, one can show that the CS action (2.7) is invariant under the following infinitesimal gauge transformations:

$$\begin{aligned}
\delta_\Lambda W^A &= D_W\rho^A,\\
\delta_\Lambda E^A &= D_W\varepsilon^A-\epsilon^{ABC}E_B\rho_C,\\
\delta_\Lambda K^A &= D_W\gamma^A-\epsilon^{ABC}\left(E_B\varepsilon_C-\rho_B K_C\right),\\
\delta_\Lambda U^A &= D_W\nu^A-\epsilon^{ABC}\left(E_B\gamma_C-\rho_B U_C\right),
\end{aligned} \tag{2.8}$$

where $\Lambda=\rho^A J_A+\varepsilon^A P_A+\gamma^A Z_A+\nu^A N_A$ is the gauge parameter.

The field equations coming from (2.7) read

$$\begin{aligned}
\delta W^A:\quad & 0=\alpha_0 R_A(W)+\alpha_1 R_A(E)\\
&+\alpha_2\left(D_W K_A-\frac{1}{2}\epsilon_{ABC}E^B E^C\right)\\
&+\alpha_3\left(D_W U_A-\epsilon_{ABC}K^B E^C\right),\\
\delta E^A:\quad & 0=\alpha_1 R_A(W)+\alpha_2 R_A(E)\\
&+\alpha_3\left(D_W K_A-\frac{1}{2}\epsilon_{ABC}E^B E^C\right),\\
\delta K^A:\quad & 0=\alpha_2 R_A(W)+\alpha_3 R_A(E),\\
\delta U^A:\quad & 0=\alpha_3 R_A(W).
\end{aligned} \tag{2.9}$$

which imply the vanishing of every curvature when $\alpha_3\neq 0$.

The study of a NR limit, as in the Maxwell and Poincaré cases, requires to introduce $U(1)$ gauge fields in order to avoid infinities and cancel divergences. Such enlargement will allow to define a proper NR limit whose NR algebra will admit non-degenerate bilinear form.

2.1. U(1) enlargements

Let us now consider a particular $U(1)$ enlargement of the generalized Maxwell algebra by adding four extra $U(1)$ one-form gauge fields to the field content as

$$A=W^A J_A+E^A P_A+K^A Z_A+U^A N_A+MY_1+SY_2+TY_3+VY_4. \tag{2.10}$$

The new relativistic algebra, [generalized Maxwell]$\oplus u(1)^4$ algebra, admits the non-vanishing components of the invariant tensor (2.6) along with

$$\begin{aligned}
\langle Y_2Y_2\rangle &= \alpha_0,\\
\langle Y_1Y_2\rangle &= \alpha_1,\\
\langle Y_2Y_3\rangle &= \langle Y_1Y_1\rangle=\alpha_2,\\
\langle Y_2Y_4\rangle &= \langle Y_1Y_3\rangle=\alpha_3.
\end{aligned} \tag{2.11}$$

Considering the gauge connection one-form (2.10) and the invariant tensor given by (2.6) and (2.11) in the general expression of the CS action (2.2), we find the following relativistic CS gravity action

$$\begin{aligned}
I_R=\int\Big[&\alpha_0\left(W^A dW_A+\frac{1}{3}\epsilon_{ABC}W^A W^B W^C+SdS\right)\\
&+\alpha_1\left(2E^A R_A(W)+2MdS\right)\\
&+\alpha_2\left(2K^A R_A(W)+E^A R_A(E)+MdM+2SdT\right)\\
&+\alpha_3\Big(2U^A R_A(W)+2E^A D_W K^A+\frac{1}{3}\epsilon^{ABC}E_A E_B E_C\\
&+2SdV+2MdT\Big)\Big].
\end{aligned} \tag{2.12}$$

In the next section, we shall see that the presence of these abelian gauge fields are essential to obtain, after a contraction procedure, a well-defined NR version of the generalized Maxwell algebra without degeneracy. Furthermore, we will show that there is a relation between the number of $U(1)$ generators required in the relativistic theory and the number of elements of the semigroup involved in the semigroup expansion method.

3. Non-relativistic generalized Maxwell Chern-Simons gravity

In this section, we shall consider the IW contraction of the previously introduced relativistic algebra, and we will obtain a NR version of the [generalized Maxwell]$\oplus u(1)^4$ algebra. Then, we will consider the construction of a NR CS action based on the aforesaid NR algebra. For this purpose, we will provide with the non-vanishing components of the invariant tensor, which are derived as an IW contraction from the relativistic invariant tensor (2.6).

3.1. Generalized Maxwellian exotic Bargmann algebra

In the previous section, we have presented an $U(1)$ enlargement of the relativistic generalized Maxwell algebra. Here, we will obtain the NR version of this algebra. To this aim, we will introduce a dimensionless parameter ξ, and we will express the relativistic generators $\{J_0, J_a, P_0, P_a, Z_0, Z_a, N_0, N_a, Y_1, Y_2, Y_3, Y_4\}$ as a linear combination of new generators involving the ξ parameter.

As in refs. [44,73], we define the IW contraction process through the identification of the relativistic generators defining the [generalized Maxwell]$\oplus u(1)^4$ algebra, with the NR generators (denoted with a tilde) as

$$\begin{aligned}
J_0&=\frac{\tilde{J}}{2}+\xi^2\tilde{S}, & J_a&=\xi\tilde{G}_a, & Y_2&=\frac{\tilde{J}}{2}-\xi^2\tilde{S},\\
P_0&=\frac{\tilde{H}}{2\xi}+\xi\tilde{M}, & P_a&=\tilde{P}_a, & Y_1&=\frac{\tilde{H}}{2\xi}-\xi\tilde{M},\\
Z_0&=\frac{\tilde{Z}}{2\xi^2}+\tilde{T}, & Z_a&=\frac{\tilde{Z}_a}{\xi}, & Y_3&=\frac{\tilde{Z}}{2\xi^2}-\tilde{T},\\
N_0&=\frac{\tilde{N}}{2\xi^3}+\frac{\tilde{V}}{\xi}, & N_a&=\frac{\tilde{N}_a}{\xi^2}, & Y_4&=\frac{\tilde{N}}{2\xi^3}-\frac{\tilde{V}}{\xi}.
\end{aligned} \tag{3.1}$$

Considering this redefinition and applying the limit $\xi\to\infty$, the contraction of the [generalized Maxwell]$\oplus u(1)^4$ algebra leads to a new NR algebra. In particular, the NR generators satisfy the following commutation relations,

$$\begin{aligned}
\left[\tilde{J},\tilde{G}_a\right]&=\epsilon_{ab}\tilde{G}_b, & \left[\tilde{G}_a,\tilde{G}_b\right]&=-\epsilon_{ab}\tilde{S}, & \left[\tilde{H},\tilde{G}_a\right]&=\epsilon_{ab}\tilde{P}_b,\\
\left[\tilde{J},\tilde{P}_a\right]&=\epsilon_{ab}\tilde{P}_b, & \left[\tilde{G}_a,\tilde{P}_b\right]&=-\epsilon_{ab}\tilde{M}, & \left[\tilde{H},\tilde{P}_a\right]&=\epsilon_{ab}\tilde{Z}_b,
\end{aligned}$$

$$
\begin{aligned}
&\left[\tilde{J},\tilde{Z}_a\right]=\epsilon_{ab}\tilde{Z}_b, \qquad \left[\tilde{P}_a,\tilde{P}_b\right]=-\epsilon_{ab}\tilde{T}, \qquad \left[\tilde{H},\tilde{Z}_a\right]=\epsilon_{ab}\tilde{N}_b,\\
&\left[\tilde{Z},\tilde{G}_a\right]=\epsilon_{ab}\tilde{Z}_b, \qquad \left[\tilde{G}_a,\tilde{Z}_b\right]=-\epsilon_{ab}\tilde{T}, \qquad \left[\tilde{Z},\tilde{P}_a\right]=\epsilon_{ab}\tilde{N}_b,\\
&\left[\tilde{J},\tilde{N}_a\right]=\epsilon_{ab}\tilde{N}_b, \qquad \left[\tilde{P}_a,\tilde{Z}_b\right]=-\epsilon_{ab}\tilde{V}, \qquad \left[\tilde{N},\tilde{G}_a\right]=\epsilon_{ab}\tilde{N}_b,\\
&\left[\tilde{G}_a,\tilde{N}_b\right]=-\epsilon_{ab}\tilde{V},
\end{aligned} \tag{3.2}
$$

where we have defined $\epsilon_{ab}\equiv\epsilon_{0ab}$, $\epsilon^{ab}\equiv\epsilon^{0ab}$, while $a=1,2$. This is a novel NR algebra which we shall call as generalized Maxwellian extended Bargmann (GMEB) algebra. From (3.2) we can see that it contains four central extensions given by $\tilde{M}$, $\tilde{S}$, $\tilde{T}$ and $\tilde{V}$, which are related to the four extra $U(1)$ generators. Let us note that the extended Bargmann algebra [35,36] can be recoverred by setting $\tilde{Z}=\tilde{Z}_a=\tilde{T}=\tilde{N}=\tilde{N}_a=\tilde{V}=0$. On the other hand, if we set $\tilde{N}=\tilde{N}_a=\tilde{V}=0$, the Maxwellian Extended Bargmann algebra is obtained [44]. As we shall see, the presence of the central charges assures to have non-degenerate invariant bilinear form.

3.2. Non-relativistic generalized Maxwell Chern-Simons action

In order to construct a CS action for the GMEB algebra we need the NR invariant tensor. The diverse components can be obtained from the contraction (3.1) of the relativistic invariant tensor (2.6). The non-vanishing components of a non-degenerate invariant tensor for the GMEB algebra are given by

$$
\begin{aligned}
&\left\langle\tilde{J}\tilde{S}\right\rangle=-\tilde{\alpha}_0,\\
&\left\langle\tilde{G}_a\tilde{G}_b\right\rangle=\tilde{\alpha}_0\delta_{ab},\\
&\left\langle\tilde{G}_a\tilde{P}_b\right\rangle=\tilde{\alpha}_1\delta_{ab},\\
&\left\langle\tilde{H}\tilde{S}\right\rangle=\left\langle\tilde{M}\tilde{J}\right\rangle=-\tilde{\alpha}_1,\\
&\left\langle\tilde{P}_a\tilde{P}_b\right\rangle=\left\langle\tilde{G}_a\tilde{Z}_b\right\rangle=\tilde{\alpha}_2\delta_{ab},\\
&\left\langle\tilde{J}\tilde{T}\right\rangle=\left\langle\tilde{H}\tilde{M}\right\rangle=\left\langle\tilde{S}\tilde{Z}\right\rangle=-\tilde{\alpha}_2,\\
&\left\langle\tilde{G}_a\tilde{N}_b\right\rangle=\left\langle\tilde{P}_a\tilde{Z}_b\right\rangle=\tilde{\alpha}_3\delta_{ab},\\
&\left\langle\tilde{J}\tilde{V}\right\rangle=\left\langle\tilde{H}\tilde{T}\right\rangle=\left\langle\tilde{M}\tilde{Z}\right\rangle=\left\langle\tilde{S}\tilde{N}\right\rangle=-\tilde{\alpha}_3,
\end{aligned} \tag{3.3}
$$

where the relativistic parameters α's have been rescaled as

$$
\alpha_0=\tilde{\alpha}_0\xi^2, \qquad \alpha_1=\tilde{\alpha}_1\xi, \qquad \alpha_2=\tilde{\alpha}_2, \qquad \alpha_3=\tilde{\alpha}_3\xi^{-1}. \tag{3.4}
$$

As in [44,73], such rescaling is done in order to have a finite NR CS action. Now we are ready to construct the aforesaid CS action. The NR one-form gauge connection $\tilde{A}$ reads

$$
\begin{aligned}
\tilde{A}=&\,\tau\tilde{H}+e^a\tilde{P}_a+\omega\tilde{J}+\omega^a\tilde{G}_a+k\tilde{Z}+k^a\tilde{Z}_a+f\tilde{N}\\
&+f^a\tilde{N}_a+m\tilde{M}+s\tilde{S}+t\tilde{T}+v\tilde{V}.
\end{aligned} \tag{3.5}
$$

The corresponding NR curvature two-form is then written as

$$
\begin{aligned}
\tilde{F}=&\,R(\tau)\tilde{H}+R^a(e^b)\tilde{P}_a+R(\omega)\tilde{J}+R^a(\omega^b)\tilde{G}_a+R(k)\tilde{Z}\\
&+R^a(k^b)\tilde{Z}_a+R(f)\tilde{N}+R^a\left(f^b\right)\tilde{N}_a\\
&+R(m)\tilde{M}+R(s)\tilde{S}+R(t)\tilde{T}+R(v)\tilde{V},
\end{aligned} \tag{3.6}
$$

where

$$
\begin{aligned}
R(\tau)&=d\tau,\\
R^a(e^b)&=de^a+\epsilon^{ac}\omega e_c+\epsilon^{ac}\tau\omega_c,\\
R(\omega)&=d\omega,\\
R^a(\omega^b)&=d\omega^a+\epsilon^{ac}\omega\omega_c,\\
R(k)&=dk,\\
R^a(k^b)&=dk^a+\epsilon^{ac}\omega k_c+\epsilon^{ac}\tau e_c+\epsilon^{ac}k\omega_c,\\
R^a\left(f^b\right)&=df^a+\epsilon^{ac}\omega f_c+\epsilon^{ac}\tau k_c+\epsilon^{ac}ke_c+\epsilon^{ac}f\omega_c,\\
R(f)&=df,\\
R(m)&=dm+\epsilon^{ac}e_a\omega_c,\\
R(s)&=ds+\frac{1}{2}\epsilon^{ac}\omega_a\omega_c,\\
R(t)&=dt+\epsilon^{ac}\omega_a k_c+\frac{1}{2}\epsilon^{ac}e_ae_c,\\
R(v)&=dv+\epsilon^{ac}\omega_a f_c+\epsilon^{ac}e_ak_c.
\end{aligned} \tag{3.7}
$$

The NR CS action invariant under the GMEB algebra can be computed by replacing the NR one-form connection (3.5) and the invariant tensor (3.3) in the general expression for the CS action in three spacetime (2.2), or by taking the NR limit directly in (2.12). In both cases, the resulting NR CS action is given by

$$
\begin{aligned}
I_{NR}=\int&\tilde{\alpha}_0\left[\omega_aR^a(\omega^b)-2sR(\omega)\right]\\
&+\tilde{\alpha}_1\left[2e_aR^a(\omega^b)-2mR(\omega)-2\tau R(s)\right]\\
&+\tilde{\alpha}_2\Big[e_aR^a\left(e^b\right)+k_aR^a\left(\omega^b\right)+\omega_aR^a\left(k^b\right)-2sR(k)\\
&-2mR(\tau)-2tR(\omega)\Big]\\
&\tilde{\alpha}_3\Big[\omega_aR^a\left(f^b\right)+f_aR^a\left(\omega^b\right)+e_aR^a\left(k^b\right)+k_aR^a\left(e^b\right)\\
&-2sR(f)-2vR(\omega)-2mR(k)-2tR(\tau)\Big].
\end{aligned} \tag{3.8}
$$

From (3.8), we see that it contains four independent sectors, each one of those proportional to an arbitrary constant $\tilde{\alpha}_i$. The first term corresponds to the NR version of the Exotic gravity [30] which is denoted as NR exotic gravity. The second term proportional to $\tilde{\alpha}_1$ reproduces the extended Bargmann gravity action [35,36], while the third term is the MEB gravity action introduced in [44]. The new gauge fields f_a, f and v, appear explicitly in the last term proportional to $\tilde{\alpha}_3$, which corresponds to the CS action for the new NR generalized Maxwell algebra. Note that the GMEB allows to include a cosmological constant term along $\tilde{\alpha}_3$.

At the level of the gauge fields one can see that the relativistic gauge fields can be expressed in terms of the NR ones as follows

$$
\begin{aligned}
&W^0=\omega+\frac{s}{2\xi^2}, \qquad W^a=\frac{\omega^a}{\xi}, \qquad S=\omega-\frac{s}{2\xi^2},\\
&E^0=\xi\tau+\frac{m}{2\xi}, \qquad E^a=e^a, \qquad M=\xi\tau-\frac{m}{2\xi},\\
&K^0=\xi^2k+\frac{t}{2}, \qquad K^a=\xi k^a, \qquad T=\xi^2k-\frac{t}{2},\\
&N^0=\xi^3f+\xi\frac{v}{2}, \qquad N^a=\xi^2f^a, \qquad V=\xi^3f-\xi\frac{v}{2}
\end{aligned} \tag{3.9}
$$

in order to have that $A=\tilde{A}$. Then considering the rescaling of the relativistic parameters as in (3.4) and considering (3.9) in the relativistic CS action (2.12), we find the NR CS action (3.8) after applying the limit $\xi\to\infty$.

As an ending remark, one could consider, as in the Maxwell case, the inclusion of three gauge fields in the relativistic generalized Maxwell algebra and then study its NR version. Although a NR limit of the [generalized Maxwell]$\oplus u(1)^3$ gravity theory could be defined, it is possible to show that the respective NR gravity theory has a degenerate bilinear form. Such feature would imply that the equations of motion from such NR theory do not determine all the dynamical fields. Then, in order to have well-defined field equations we need non-degenerate invariant tensor which requires to consider a CS action based on the [generalized Maxwell]$\oplus u(1)^4$ algebra as the relativistic gravity theory. In particular, we have that the field equations of the GMEB theory are given by the vanishing of each curvature (3.7).

4. Generalized family of non-relativistic algebras and semigroup expansion

The expansion of a Lie algebra is a method that consists in finding a new (bigger) Lie algebra $\mathfrak{G}$, following a series of well-defined steps from an original Lie algebra $\mathfrak{g}$. This procedure was first introduced by [74] and later studied in [75,76]. It basically consists in performing a rescaling by a real parameter λ of some of the coordinates of the Lie group g^i, $i = 1, \ldots, \dim \mathfrak{g}$, and then expanding the Maurer-Cartan (MC) one-forms in powers of the parameter λ. An alternative expansion method, called as S-expansion, was subsequently introduced in [62]. The S-expansion procedure consists in obtaining a new Lie algebra $\mathfrak{G} = S \times \mathfrak{g}$, by combining the elements of a semigroup S with the structure constants of a Lie algebra $\mathfrak{g}$. This approach is entirely based on operations performed on the algebra generators, whereas the aforesaid power series expansion is carried out on the MC one-forms. Another point in which both procedures differ, lies in the fact that the S-expansion is defined on the Lie algebra $\mathfrak{g}$ without mentioning the group manifold, while the power series expansion is based on the rescaling of the group coordinates. Remarkably, the semigroup expansion method can reproduce the MC forms power series expansion for a particular choice of the semigroup S. On the other hand, one of the advantage of working with the S-expansion is that it not only provides the commutation relations of the expanded algebra, but also allows to compute the non-vanishing components of the invariant tensor of the expanded algebra in terms of the invariant tensor for the original algebra.

In this section, following the procedure used in [67,73], we review a generalized family of NR algebras by considering the semigroup expansion method. Then, we extend the results obtained in [67] by showing that the S-expansion not only allows to obtain expanded NR algebras but also provides with their relativistic versions and the appropriate rescaling of the generators allowing a proper NR limit.

Here, we consider the Nappi-Witten algebra [77,78] as the original algebra $\mathfrak{g}$, which can be seen as a central extension of the homogeneous part of the Galilei algebra. The Nappi-Witten algebra is spanned by the set of generators $\left\{\tilde{J}, \tilde{G}_a, \tilde{S}\right\}$ which satisfy the following non-vanishing commutation relations,

$$
\begin{aligned}
\left[\tilde{J}, \tilde{G}_a\right] &= \epsilon_{ab}\tilde{G}_b, \\
\left[\tilde{G}_a, \tilde{G}_b\right] &= -\epsilon_{ab}\tilde{S},
\end{aligned}
\tag{4.1}
$$

where $\tilde{J}$ are spatial rotations, $\tilde{G}_a$ are Galilean boosts and $\tilde{S}$ is a central charge. Such algebra can be obtained as a contraction of an $U(1)$-enlargement of the Lorentz algebra through the identification of the [Lorentz]$\oplus u(1)$ generators with the Nappi-Witten ones as

$$
J_0 = \frac{\tilde{J}}{2} + \xi^2\tilde{S}, \qquad J_a = \xi\tilde{G}_a, \qquad Y = \frac{\tilde{J}}{2} - \xi^2\tilde{S}. \tag{4.2}
$$

Then the Nappi-Witten algebra is obtained after applying the limit $\xi \to \infty$. One can see that the non-vanishing components of a non-degenerate invariant tensor of the Nappi-Witten algebra read

$$
\begin{aligned}
\left\langle \tilde{J}\tilde{S} \right\rangle &= -1, \\
\left\langle \tilde{G}_a\tilde{G}_b \right\rangle &= \delta_{ab}.
\end{aligned}
\tag{4.3}
$$

In what follows, we shall review the family of NR symmetries obtained as $S_E^{(N)}$-expansions of the Nappi-Witten algebra [67]. We shall denote such family as generalized extended Bargmann GEB$^{(N)}$ algebra. Interestingly, we will see that the extended Bargmann, the MEB and the GMEB algebra previously introduced in the previous section are particular cases of the GEB$^{(N)}$ algebra. Furthermore, we will show that the same semigroup can be used at the relativistic level providing with the relativistic version of the GEB$^{(N)}$ algebra. Before approaching the family of NR algebras and their respective NR CS actions, we first show that the GMEB algebra can alternatively be obtained by expanding the Nappi-Witten algebra.

4.1. Generalized Maxwellian exotic Bargmann gravity from semigroup expansion

Let $S_E^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be the relevant semigroup whose elements satisfy the following multiplication law

$$
\lambda_\alpha\lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta < 4, \\ \lambda_4 & \text{if } \alpha+\beta \geq 4, \end{cases} \tag{4.4}
$$

being $\lambda_4 = 0_S$ the zero element, such that $0_S\lambda_\alpha = 0_S$. An expanded algebra can be obtained after performing a 0_S-reduction to the $S_E^{(3)}$-expansion of the Nappi-Witten algebra following the definitions of [62]. The expanded generators $\{\tilde{J}, \tilde{G}_a, \tilde{H}, \tilde{P}_a, \tilde{Z}, \tilde{Z}_a, \tilde{N}, \tilde{N}_a, \tilde{S}, \tilde{M}, \tilde{T}, \tilde{V}\}$ are related to the Nappi-Witten ones through the semigroup elements as

λ_3	$\tilde{N}$	$\tilde{N}_a$	$\tilde{V}$
λ_2	$\tilde{Z}$	$\tilde{Z}_a$	$\tilde{T}$
λ_1	$\tilde{H}$	$\tilde{P}_a$	$\tilde{M}$
λ_0	$\tilde{J}$	$\tilde{G}_a$	$\tilde{S}$
	$\tilde{J}$	$\tilde{G}_a$	$\tilde{S}$

(4.5)

Let us note that the 0_S-reduction condition implies that $0_S T_A = 0$, with T_A being a generator of the original algebra. Using the commutation relations of the Nappi-Witten algebra (4.1) and the multiplication law of the semigroup (4.4), one can show that the expanded generators satisfy the GMEB algebra (3.2) previously introduced.

Interestingly, the S-expansion procedure can also provide with the non-vanishing components of the invariant tensor for the expanded algebra. Indeed, considering the Theorem VII of [62], it is possible to express the invariant tensor of the expanded algebra in terms of the Nappi-Witten ones (4.3) as

$$
\left\langle T_A^{(\nu)} T_B^{(\mu)} \right\rangle_{S_E^{(3)}\times\mathfrak{g}} = \alpha_\gamma K_{\nu\mu}{}^{\gamma} \left\langle T_A T_B \right\rangle_{\mathfrak{g}}, \tag{4.6}
$$

where $T_A^{(\nu)}$ are the corresponding expanded generators, α_γ are arbitrary constants and $K_{\nu\mu}{}^{\gamma}$ is the 2-selector for $S_E^{(3)}$ satisfying

$$
K_{\nu\mu}{}^{\gamma} = \begin{cases} 1, & \text{when } \gamma = \nu+\mu \text{ and } \nu+\mu < 4 \\ 0, & \text{otherwise.} \end{cases} \tag{4.7}
$$

Then one can see that the non-vanishing components of the invariant tensor for the expanded algebra are those of the GMEB algebra given by (3.3). Thus, the $S_E^{(3)}$-expansion of the Nappi-Witten algebra provides not only with the commutation relations of the GMEB algebra but also with its invariant tensor which is the crucial ingredient for the construction of a CS action.

4.2. Generalized extended Bargmann family

A family of generalized extended Bargmann algebra can be obtained by S-expanding the Nappi-Witten algebra (4.1) considering $S_E^{(N)} = \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_N, \lambda_{N+1}\}$ as the relevant semigroup [67]. In particular, the elements of the semigroup $S_E^{(N)}$ satisfy the following multiplication law

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta} & \text{if } \alpha+\beta < N+1 \\ \lambda_{N+1} & \text{if } \alpha+\beta \geq N+1 \end{cases} \tag{4.8}$$

with $\lambda_{N+1} \equiv 0_S$ being the zero element of the semigroup such that $0_S \lambda_\alpha = 0_S$. An expanded algebra is found by performing a 0_S-reduction to the $S_E^{(N)}$-expansion of the Nappi-Witten algebra. The expanded NR algebra is spanned by the set of the generators $\left\{\tilde{J}^{(i)}, \tilde{G}_a^{(i)}, \tilde{S}^{(i)}\right\}$ with $i = 0, \dots, N$. Such NR generators are related to the Nappi-Witten ones through the semigroup elements as

$$\tilde{J}^{(i)} = \lambda_i \tilde{J}, \qquad \tilde{G}_a^{(i)} = \lambda_i \tilde{G}_a, \qquad \tilde{S}^{(i)} = \lambda_i \tilde{S}. \tag{4.9}$$

Then, using the multiplication law of the semigroup (4.8) and the original commutators of the Nappi-Witten algebra (4.1), one can show that the expanded NR algebra satisfy the following non-vanishing commutation relations

$$\begin{aligned} \left[\tilde{J}^{(i)}, \tilde{G}_a^{(j)}\right] &= \epsilon_{ab} \tilde{G}_b^{(i+j)}, \\ \left[\tilde{G}_a^{(i)}, \tilde{G}_b^{(j)}\right] &= -\epsilon_{ab} \tilde{S}^{(i+j)}, \end{aligned} \tag{4.10}$$

for $i + j < N + 1$. The expanded NR algebra can be seen as a generalization of the extended Bargmann algebra and is denoted as $\text{GEB}^{(N)}$ algebra.

Interestingly, one can show that the extended Bargmann algebra [35] is obtained for $N = 1$. Indeed, we have that the $\text{GEB}^{(1)}$ algebra, which is given by

$$\begin{aligned} \left[\tilde{J}^{(0)}, \tilde{G}_a^{(0)}\right] &= \epsilon_{ab} \tilde{G}_b^{(0)}, & \left[\tilde{G}_a^{(0)}, \tilde{G}_b^{(0)}\right] &= -\epsilon_{ab} \tilde{S}^{(0)}, \\ \left[\tilde{J}^{(0)}, \tilde{G}_a^{(1)}\right] &= \epsilon_{ab} \tilde{G}_b^{(1)}, & \left[\tilde{G}_a^{(0)}, \tilde{G}_b^{(1)}\right] &= -\epsilon_{ab} \tilde{S}^{(1)}, \\ \left[\tilde{J}^{(1)}, \tilde{G}_a^{(0)}\right] &= \epsilon_{ab} \tilde{G}_b^{(1)}, \end{aligned} \tag{4.11}$$

corresponds to the extended Bargmann algebra by identifying the generators as

$$\begin{aligned} &\tilde{J}^{(0)} = \tilde{J}, \quad \tilde{G}_a^{(0)} = \tilde{G}_a, \quad \tilde{S}^{(0)} = \tilde{S}, \\ &\tilde{J}^{(1)} = \tilde{H}, \quad \tilde{G}_a^{(1)} = \tilde{P}_a, \quad \tilde{S}^{(1)} = \tilde{M}. \end{aligned} \tag{4.12}$$

On the other hand, for $N = 2$, one can see that the $\text{GEB}^{(2)}$ algebra corresponds to the MEB algebra introduced in [44], which is given by (4.11) along with

$$\begin{aligned} \left[\tilde{J}^{(0)}, \tilde{G}_a^{(2)}\right] &= \epsilon_{ab} \tilde{G}_b^{(2)}, & \left[\tilde{G}_a^{(0)}, \tilde{G}_b^{(2)}\right] &= -\epsilon_{ab} \tilde{S}^{(2)}, \\ \left[\tilde{J}^{(2)}, \tilde{G}_a^{(0)}\right] &= \epsilon_{ab} \tilde{G}_b^{(2)}, & \left[\tilde{G}_a^{(1)}, \tilde{G}_b^{(1)}\right] &= -\epsilon_{ab} \tilde{S}^{(2)}, \\ \left[\tilde{J}^{(1)}, \tilde{G}_a^{(1)}\right] &= \epsilon_{ab} \tilde{G}_b^{(1)}, \end{aligned} \tag{4.13}$$

where one can consider the identification (4.12) together with

$$\tilde{J}^{(2)} = \tilde{Z}, \quad \tilde{G}_a^{(2)} = \tilde{Z}_a, \quad \tilde{S}^{(2)} = \tilde{T}. \tag{4.14}$$

The GMEB algebra presented here can also be seen as a particular case of the $\text{GEB}^{(N)}$ algebra. In fact, as we have previously shown, the GMEB algebra appears as a $S_E^{(3)}$-expansion of the Nappi-Witten algebra.

Interestingly, the $\text{GEB}^{(N)}$ algebra is the respective NR version of $U(1)$-enlargements of the so-called $\mathfrak{B}_{N+2}$ algebra introduced in [54,55],

$$\mathfrak{B}_{N+2} \oplus u(1)^{N+1} = \begin{cases} [\text{Poincaré}] \oplus u(1)^2 \\ [\text{Maxwell}] \oplus u(1)^3 \\ [\text{Gen. Maxwell}] \oplus u(1)^4 \\ \vdots \\ \mathfrak{B}_{N+1} \oplus u(1)^N \\ \mathfrak{B}_{N+2} \oplus u(1)^{N+1} \end{cases}$$

$$\xrightarrow{\text{NR limit}} \quad \text{GEB}^{(N)} = \begin{cases} \text{Extended Bargmann} \\ \text{MEB} \\ \text{GMEB} \\ \vdots \\ \text{GEB}^{(N-1)} \\ \text{GEB}^{(N)} \end{cases}$$

Let us note that the Poincaré, Maxwell and generalized Maxwell algebras are the $\mathfrak{B}_3$, $\mathfrak{B}_4$ and $\mathfrak{B}_5$ algebras, respectively. Moreover, analogously to [73], the $S_E^{(N)}$ semigroup used to obtain the $\text{GEB}^{(N)}$ algebra is the same used to find the $\mathfrak{B}_{N+2}$ algebra from the Lorentz algebra. Such particularity also appears for infinite-dimensional (super)algebras [79–81] and algebras coupled to spin-3 [59]. It is interesting to point out that the number of additional $U(1)$ generators appearing in the relativistic algebra is related to the $N + 1$ elements of the semigroup $S_E^{(N)}$. This is due to the fact that the $\mathfrak{B}_{N+2} \oplus u(1)^{N+1}$ algebra can be recovered as a $S_E^{(N)}$-expansion of the [Lorentz]$\oplus u(1)$ algebra.

Let us note that the S-expansion procedure can also provides with the proper NR limit (3.1) leading to the $\text{GEB}^{(N)}$ algebra in terms of the contraction process (4.2) allowing to obtain the Nappi-Witten algebra. As we have previously mentioned, the Nappi-Witten algebra can be found as an IW contraction of the [Lorentz]$\oplus u(1)$ algebra. Interestingly, the identification of the relativistic generators defining $\mathfrak{B}_{N+2} \oplus u(1)^{N+1}$ algebra, with the NR generators (denoted with a tilde) can be defined as

$$\begin{aligned} J_0^{(i)} &= \frac{\tilde{J}^{(i)}}{2\xi^i} + \xi^{2-i} \tilde{S}^{(i)}, \qquad J_a^{(i)} = \xi^{1-i} \tilde{G}_a^{(i)}, \\ Y^{(i)} &= \frac{\tilde{J}^{(i)}}{2} - \xi^{2-i} \tilde{S}^{(i)}, \end{aligned} \tag{4.15}$$

with $i = 0, 1, \dots, N$. Here, the relativistic generators are related to the [Lorentz]$\oplus u(1)$ ones through the elements of $S_E^{(N)}$ as

$$J_0^{(i)} = \lambda_i J_0, \qquad J_a^{(i)} = \lambda_i J_a, \qquad Y^{(i)} = \lambda_i Y. \tag{4.16}$$

Thus, the semigroup $S_E^{(N)}$ leads to the proper $U(1)$-enlargement of the relativistic theory which leads to a non-degenerate and finite NR gravity theory. One can check that for $N = 1, 2, 3$ we recover the contraction process for the extended Bargmann, MEB and GMEB, respectively.

On the other hand, as we have previously noted, an additional advantage of the S-expansion method is that it provides with the invariant tensors of the expanded algebra which are crucial for the construction of a CS action. Thus, following Theorem VII of [62], one can show that the non-vanishing components of the invariant tensor of the $\text{GEB}^{(N)}$ algebra is given by

$$\begin{aligned} \left\langle \tilde{J}^{(i)} \tilde{S}^{(i)} \right\rangle &= -\tilde{\alpha}_{i+j}, \\ \left\langle \tilde{G}_a^{(i)} \tilde{G}_b^{(j)} \right\rangle &= \tilde{\alpha}_{i+j} \, \delta_{ab}, \end{aligned} \tag{4.17}$$

for $i+j<N+1$. Then the NR CS action based on the GEB$^{(N)}$ algebra expressed in term of the gauge connection one-form $A=\omega^{(i)}\tilde{J}^{(i)}+\omega^{a(i)}\tilde{G}_a^{(i)}+s^{(i)}S^{(i)}$ is given by

$$I_{NR}=\int\tilde{\alpha}_i\left[\omega_a^{(j)}d\omega^{a(k)}\delta^i_{j+k}+\epsilon^{ac}\omega_a^{(j)}\omega^{(k)}\omega_c^{(l)}\delta^i_{j+k+l}\right.$$
$$\left.-2s^{(j)}d\omega^{(k)}\delta^i_{j+k}\right],\tag{4.18}$$

with $i=0,1,\ldots,N$. The NR CS gravity action contains i independent sectors each one invariant under the GEB$^{(N)}$ algebra. In particular, the term proportional to α_0 corresponds to the NR exotic gravity action. The terms proportional to α_1 and α_2 are the extended Bargmann gravity [35,36] and MEB gravity [44] actions, respectively. The NR CS action for the GMEB algebra previously defined appears along α_3. On the other hand, for $3<i\leq N$, the additional gauge fields related to $J^{(i)}$, $G_a^{(i)}$ and $S^{(i)}$ appear explicitly along $\tilde{\alpha}_i$, corresponding to the respective NR CS action for the GEB$^{(i)}$ algebra. Let us notice that a general expression for the CS action based on an expanded Nappi-Witten algebra for a semigroup S has been presented in [67].

5. Conclusions

In this paper we have presented a generalization of the Maxwellian extended Bargmann gravity introduced in [44]. We have explicitly shown that this NR symmetry, that we have denoted as GMEB, can be obtained as an IW contraction of the [generalized Maxwell]$\oplus u(1)^4$ algebra. To this end, we first presented the CS gravity action invariant under the generalized Maxwell algebra. Then, we constructed an $U(1)$-enlargement which is required to have a well-defined NR limit. Interestingly, the GMEB gravity theory contains the MEB and the extended Bargmann theories as sub-cases. The GMEB algebra belongs to a generalized family of NR algebras which can be obtained by expanding the Nappi-Witten algebra [67]. Here, we have shown that the expansion procedure based on semigroups is a powerful tool in the NR context since it provides not only with the commutation relations and invariant tensor of the expanded NR algebras, but also with the respective relativistic algebra required to obtain non-degenerate finite NR gravity theories.

Our results could be useful in the presence of supersymmetry. The construction of proper NR supergravity models are nontrivial and have only recently been approached. In particular the respective NR superalgebras have mainly been constructed by hand in three spacetime dimensions [12,21,25,32,35,43]. The expansion method considered here could not only be used as an alternative and straightforward way to obtain known and new NR superalgebras but also to construct NR supergravity actions. Furthermore, it could bring invaluable information about the relativistic versions and the respective NR limits as in our case. Let us notice that the Lie algebra expansion method using the Maurer-Cartan equations [75,76] has also been used in the NR context with diverse interesting results [82–85].

Let us note that, as was shown in [67], the GEB$^{(N)}$ algebra appears as a IW contraction of another family of NR algebras denoted as generalized Newton-Hooke. Then, another aspect that deserved to be explored is the derivation of this generalized Newton-Hooke algebra as an IW contraction of a family of relativistic algebras. One could conjecture that, similarly to our results, the semigroup procedure could be useful to elucidate the appropriate $U(1)$-enlargement of the relativistic family.

Regarding the relativistic generalized Maxwell gravity theory, it would be interesting to explore its general solution and asymptotic symmetry. In particular, one could analyze the influence of the additional gauge field in the vacuum energy and angular momentum, and compare them to those of General Relativity [86,87] and usual Maxwell theory [51]. One could expect that the asymptotic structure is given by the infinite-dimensional enhancement of the $\mathfrak{B}_5$ algebra introduced in [79].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] K. Kuchar, Gravitation, geometry, and nonrelativistic quantum theory, Phys. Rev. D 22 (1980) 1285.
[2] C. Duval, H.P. Kunzle, Minimal gravitational coupling in the Newtonian theory and the covariant Schrödinger equation, Gen. Relativ. Gravit. 16 (1984) 333.
[3] C. Duval, G. Burdet, H.P. Kunzle, M. Perrin, Bargmann structures and Newton-Cartan theory, Phys. Rev. D 31 (1985) 1841.
[4] C. Duval, G.W. Gibbons, P. Horvathy, Celestial mechanics, conformal structures and gravitational waves, Phys. Rev. D 43 (1991) 3907, arXiv:hep-th/0512188.
[5] C. Duval, On Galilean isometries, Class. Quantum Gravity 10 (1993) 2217, arXiv:0903.1641 [math-ph].
[6] R. De Pietri, L. Lusanna, M. Pauri, Standard and generalized Newtonian gravities as 'gauge' theories of the extended Galilei group. I. The standard theory, Class. Quantum Gravity 12 (1995) 219, arXiv:gr-qc/9405046.
[7] R. De Pietri, L. Lusanna, M. Pauri, Standard and generalized Newtonian gravities as 'gauge' theories of the extended Galilei group. II. Dynamical three space theories, Class. Quantum Gravity 12 (1995) 255, arXiv:gr-qc/9405047.
[8] P. Hořava, Quantum gravity at a Lifshitz point, Phys. Rev. D 79 (2009) 084008, arXiv:0901.3775 [hep-th].
[9] C. Duval, P.A. Horvathy, Non-relativistic conformal symmetries and Newton-Cartan structures, J. Phys. A 42 (2009) 465206, arXiv:0904.0531 [math-ph].
[10] R. Andringa, E. Bergshoeff, S. Panda, M. de Roo, Newtonian gravity and the Bargmann algebra, Class. Quantum Gravity 28 (2011) 105011, arXiv:1011.1145 [hep-th].
[11] R. Andringa, E. Bergshoeff, J. Gomis, M. de Roo, 'Stringly' Newton-Cartan gravity, Class. Quantum Gravity 29 (2012) 235020, arXiv:1206.5176 [hep-th].
[12] R. Andringa, E.A. Bergshoeff, J. Rosseel, E. Sezgin, 3D Newton-Cartan supergravity, Class. Quantum Gravity 30 (2013) 205005, arXiv:1305.6737 [hep-th].
[13] R. Banerjee, A. Mitra, P. Mukherjee, Localisation of the Galilean symmetry and dynamical realisation of Newton-Cartan geometry, Class. Quantum Gravity 32 (2015) 045010, arXiv:1407.3617 [hep-th].
[14] X. Bekaert, K. Morand, Connections and dynamical trajectories in generalised Newton-Cartan gravity I. An intrinsic view, J. Math. Phys. 57 (2016) 022507, arXiv:1412.8213 [hep-th].
[15] X. Bekaert, K. Morand, Connections and dynamical trajectories in generalised Newton-Cartan gravity II. An ambient perspective, J. Math. Phys. 59 (2018) 072503, arXiv:1505.03739 [hep-th].
[16] E. Bergshoeff, J. Rosseel, T. Zojer, Newton-Cartan supergravity with torsion and Schrödinger supergravity, J. High Energy Phys. 1511 (2015) 180, arXiv:1509.04527 [hep-th].
[17] R. Banerjee, P. Mukherjee, Torsional Newton-Cartan geometry from Galilean gauge theory, Class. Quantum Gravity 33 (2016) 225013, arXiv:1604.06893 [gr-qc].
[18] E. Bergshoeff, A. Chatzistavrakidis, L. Romano, J. Rosseel, Newton-Cartan gravity and Torsion, J. High Energy Phys. 1710 (2017) 194, arXiv:1708.05414 [hep.th].
[19] D. Hansen, J. Hartong, N.A. Obers, Action principle for Newtonian gravity, Phys. Rev. Lett. 122 (2019) 061106, arXiv:1807.04765 [hep-th].
[20] R. Banerjee, P. Mukherjee, Galilean gauge theory from Poincare gauge theory, Phys. Rev. D 98 (2018) 124021, arXiv:1810.03902 [gr-qc].
[21] N. Ozdemir, M. Ozkan, O. Tunca, U. Zorba, Three-dimensional extended Newtonian (super)gravity, J. High Energy Phys. 1905 (2019) 130, arXiv:1903.09377 [hep-th].
[22] J. Matulich, S. Prohazka, J. Salzer, Limits of three-dimensional gravity and metric kinematical Lie algebras in any dimension, J. High Energy Phys. 07 (2019) 118, arXiv:1903.09165 [hep-th].

[23] D. Chernyavsky, D. Sorokin, Three-dimensional (higher-spin) gravities with extended Schrödinger and l-conformal Galilean symmetries, J. High Energy Phys. 07 (2019) 156, arXiv:1905.13154 [hep-th].
[24] P. Concha, L. Ravera, E. Rodríguez, Three-dimensional exotic Newtonian gravity with cosmological constant, Phys. Lett. B 804 (2020) 135392, arXiv:1912.02836 [hep-th].
[25] P. Concha, L. Ravera, E. Rodríguez, Three-dimensional Maxwellian extended Bargmann supergravity, J. High Energy Phys. 04 (2020) 051, arXiv:1912.09477 [hep-th].
[26] J. Gomis, A. Kleinschmidt, J. Palmkvist, P. Salgado-Rebolledo, Newton-Hooke/Carrollian expansions of (A)dS and Chern-Simons gravity, J. High Energy Phys. 2002 (2020) 009, arXiv:1912.07564 [hep-th].
[27] E. Bergshoeff, J. Gomis, P. Salgado-Rebolledo, Non-relativistic limits and three-dimensional coadjoint Poincare gravity, arXiv:2001.11790 [hep-th].
[28] M. Ergen, E. Hamamci, D. Van den Bleeken, Oddity in nonrelativistic, strong gravity, arXiv:2002.02688 [gr-qc].
[29] A. Achucarro, P.K. Townsend, A Chern-Simons action for three-dimensional anti-de Sitter supergravity theories, Phys. Lett. B 180 (1986) 89.
[30] E. Witten, (2+1)-dimensional gravity as an exactly soluble system, Nucl. Phys. B 311 (1988) 46.
[31] J. Zanelli, Lecture notes on Chern-Simons (super-)gravities, second edition, arXiv:hep-th/0502193, February 2008.
[32] E. Bergshoeff, J. Rosseel, T. Zojer, Newton-Cartan (super)gravity as a non-relativistic limit, Class. Quantum Gravity 32 (2015) 205003, arXiv:1505.02095 [hep-th].
[33] J. Gomis, H. Ooguri, Nonrelativistic closed string theory, J. Math. Phys. 42 (2001) 3127, arXiv:hep-th/0009181.
[34] A. Barducci, R. Casalbuoni, J. Gomis, Non-relativistic spinning particle in a Newton-Cartan background, J. High Energy Phys. 01 (2018) 002, arXiv:1710.10970 [hep-th].
[35] E.A. Bergshoeff, J. Rosseel, Three-dimensional extended Bargmann supergravity, Phys. Rev. Lett. 116 (2016) 251601, arXiv:1604.08042 [hep-th].
[36] J. Hartong, Y. Lei, N.A. Obers, Nonrelativistic Chern-Simons theories and three-dimensional Horava-Lifshitz gravity, Phys. Rev. D 94 (2016) 065027, arXiv:1604.08054 [hep-th].
[37] G. Papageorgiou, B.J. Schroers, Galilean quantum gravity with cosmological constant and the extended q-Heisenberg algebra, J. High Energy Phys. 11 (2010) 020, arXiv:1008.0279 [hep-th].
[38] O. Arratia, M.A. Martin, M.A. Olmo, Classical systems and representations of $(2+1)$ newton-Hooke symmetries, arXiv:math-ph/9903013.
[39] Y.H. Gao, Symmetries, matrices, and de Sitter gravity, arXiv:hep-th/0107067.
[40] G.W. Gibbons, C.E. Patricot, Newton-Hooke spacetimes, Hpp-waves and the cosmological constant, Class. Quantum Gravity 20 (2003) 5225, arXiv:hep-th/0308200.
[41] J. Brugues, J. Gomis, K. Kamimura, Newton-Hooke algebras, non-relativistic branes and generalized pp-wave metrics, Phys. Rev. D 73 (2006) 085011, arXiv:hep-th/0603023.
[42] P.D. Alvarez, J. Gomis, K. Kamimura, M.S. Plyushchay, $(2+1)$D exotic Newton-Hooke symmetry, duality and projective phase, Ann. Phys. 322 (2007) 1556, arXiv:hep-th/0702014.
[43] N. Ozdemir, M. Ozkan, U. Zorba, Three-dimensional extended Lifshitz, Schrödinger and Newton-Hooke supergravity, J. High Energy Phys. 1911 (2019) 052, arXiv:1909.10745 [hep-th].
[44] L. Avilés, E. Frodden, J. Gomis, D. Hidalgo, J. Zanelli, Non-relativistic Maxwell Chern-Simons gravity, J. High Energy Phys. 1805 (2018) 047, arXiv:1802.08453 [hep-th].
[45] R. Schrader, The Maxwell group and the quantum theory of particles in classical homogeneous electromagnetic fields, Fortschr. Phys. 20 (1972) 701.
[46] H. Bacry, P. Combe, J.L. Richard, Group-theoretical analysis of elementary particles in an external electromagnetic field. 1. The relativistic particle in a constant and uniform field, Nuovo Cimento A 67 (1970) 267.
[47] J. Gomis, A. Kleinschmidt, On free Lie algebras and particles in electro-magnetic fields, J. High Energy Phys. 07 (2017) 085, arXiv:1705.05854 [hep-th].
[48] P. Salgado, R.J. Szabo, O. Valdivia, Topological gravity and transgression holography, Phys. Rev. D 89 (2014) 084077, arXiv:1401.3653 [hep-th].
[49] S. Hoseinzadeh, A. Rezaei-Aghdam, (2+1)-dimensional gravity from Maxwell and semisimple extension of the Poincaré gauge symmetric models, Phys. Rev. D 90 (2014) 084008, arXiv:1402.0320 [hep-th].
[50] P.K. Concha, O. Fierro, E.K. Rodríguez, P. Salgado, Chern-Simons supergravity in $D=3$ and Maxwell superalgebra, Phys. Lett. B 750 (2015) 117, arXiv:1507.02335 [hep-th].
[51] P. Concha, N. Merino, O. Miskovic, E. Rodríguez, P. Salgado-Rebolledo, O. Valdivia, Asymptotic symmetries of three-dimensional Chern-Simons gravity for the Maxwell algebra, J. High Energy Phys. 1810 (2018) 079, arXiv:1805.08834 [hep-th].
[52] S. Bansal, D. Sorokin, Can Chern-Simons or Rarita-Schwinger ber a Volkov-Akulov Goldstone?, J. High Energy Phys. 07 (2018) 106, arXiv:1806.05945 [hep-th].
[53] D. Chernyavsky, N. Sadik Deger, D. Sorokin, Spontaneously broken 3d Hietarinta-Maxwell Chern-Simons theory and minimal massive gravity, arXiv:2002.07592 [hep-th].
[54] J.D. Edelstein, M. Hassaine, R. Troncoso, J. Zanelli, Lie-algebra expansions, Chern-Simons theories and the Einstein-Hilbert Lagrangian, Phys. Lett. B 640 (2006) 278, arXiv:hep-th/0605174.
[55] F. Izaurieta, E. Rodríguez, P. Minning, P. Salgado, A. Perez, Standard general relativity from Chern-Simons gravity, Phys. Lett. B 678 (2009) 213, arXiv:0905.2187 [hep-th].
[56] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez, P. Salgado, Even-dimensional general relativity from born-infeld gravity, Phys. Lett. B 725 (2013) 419, arXiv:1309.0062 [hep-th].
[57] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez, P. Salgado, Chern-Simons and born-infeld gravity theories and Maxwell algebras type, Eur. Phys. J. C 74 (2014) 2741, arXiv:1402.0023 [hep-th].
[58] P.K. Concha, D.M. Peñafiel, E.K. Rodríguez, P. Salgado, Generalized Poincaré algebras and Lovelock-Cartan gravity theory, Phys. Lett. B 742 (2015) 310, arXiv:1405.7078 [hep.th].
[59] R. Caroca, P. Concha, O. Fierro, E. Rodríguez, P. Salgado-Rebolledo, Generalized Chern-Simons higher-spin gravity theories in three dimensions, Nucl. Phys. B 934 (2018) 240, arXiv:1712.09975 [hep-th].
[60] E. Inönü, E.P. Wigner, On the contraction of groups and their representations, Proc. Natl. Acad. Sci. USA 39 (1953) 510.
[61] E. Weimar-Woods, Contractions, generalized Inönü-Wigner contractions and deformations of finite-dimensional Lie algebras, Rev. Mod. Phys. 12 (2000) 1505.
[62] F. Izaurieta, E. Rodríguez, P. Salgado, Expanding Lie (super)algebras through Abelian semigroups, J. Math. Phys. 47 (2006) 123512, arXiv:hep-th/0606215.
[63] R. Caroca, I. Kondrashuk, N. Merino, F. Nadal, Bianchi spaces and their 3-dimensional isometries as S-expansions of 2-dimensional isometries, J. Phys. A 46 (2013) 225201, arXiv:1104.3541 [math-ph].
[64] L. Andrianopoli, N. Merino, F. Nadal, M. Trigiante, General properties of the expansion methods of Lie algebras, J. Phys. A 46 (2013) 365204, arXiv:1308.4832 [gr-qc].
[65] M. Artebani, R. Caroca, M.C. Ipinza, D.M. Peñafiel, P. Salgado, Geometrical aspects of the Lie algebra S-expansion procedure, J. Math. Phys. 57 (2016) 023516, arXiv:1602.04525 [math-ph].
[66] M.C. Ipinza, F. Lingua, D.M. Peñafiel, L. Ravera, An analytic method for S-expansion involving resonance and reduction, Fortschr. Phys. 64 (2016) 854, arXiv:1609.05042 [hep-th].
[67] D.M. Peñafiel, P. Salgado-Rebolledo, Non-relativistic symmetries in three space-time dimensions and the Nappi-Witten algebra, Phys. Lett. B 798 (2019) 135005, arXiv:1906.02161 [hep-th].
[68] D.V. Soroka, V.A. Soroka, Semi-simple extension of the (super) Poincaré algebra, Adv. High Energy Phys. 2009 (2009) 234147, arXiv:hep-th/0605251.
[69] J. Díaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodriguez, P. Salgado, O. Valdivia, A generalized action for $(2+1)$-dimensional Chern-Simons gravity, J. Phys. A, Math. Theor.45 (2012) 255207, arXiv:1311.2215 [gr-qc].
[70] P. Salgado, S. Salgado, $\mathfrak{so}(D-1,1) \oplus \mathfrak{so}(D-1,2)$ algebras and gravity, Phys. Lett. B 728 (2014) 5–10.
[71] P. Concha, D.M. Peñafiel, E. Rodríguez, On the Maxwell supergravity and flat limit in 2+1 dimensions, Phys. Lett. B 785 (2018) 247, arXiv:1807.00194 [hep-th].
[72] P. Concha, $\mathcal{N}$-extended Maxwell supergravities as Chern-Simons theories in three spacetime dimensions, Phys. Lett. B 792 (2019) 290, arXiv:1903.03081 [hep-th].
[73] P. Concha, E. Rodríguez, Non-relativistic gravity theory based on an enlargement of the extended Bargmann algebra, J. High Energy Phys. 07 (2019) 085, arXiv:1906.00086 [hep-th].
[74] M. Hatsuda, M. Sakaguchi, Wess-Zumino term for the AdS superstring and generalized Inönü-Wigner contraction, Prog. Theor. Phys. 109 (2003) 853–867, arXiv:hep-th/0106114.
[75] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Generating Lie and gauge free differential (super)algebras by expanding Maurer-Cartan forms and Chern-Simons supergravity, Nucl. Phys. B 662 (2003) 185, arXiv:hep-th/0212347.
[76] J.A. de Azcárraga, J.M. Izquierdo, M. Picón, O. Varela, Expansions of algebras and superalgebras and some applications, Int. J. Theor. Phys. 46 (2007) 2738, arXiv:hep-th/0703017.
[77] C.R. Nappi, E. Witten, A WZW model based on a nonsemisimple group, Phys. Rev. Lett. 71 (1993) 3751, arXiv:hep-th/9310112.
[78] J. Figueroa-O'Farrill, S. Stanciu, More D-branes in the Nappi-Witten background, J. High Energy Phys. 01 (2000) 024, arXiv:hep-th/9909164.
[79] R. Caroca, P. Concha, E. Rodríguez, P. Salgado-Rebolledo, Generalizing the $\mathfrak{bms}_3$ and 2D-conformal algebras by expanding the Virasoro algebra, Eur. Phys. J. C 78 (2018) 262, arXiv:1707.07209 [hep-th].
[80] R. Caroca, P. Concha, O. Fierro, E. Rodríguez, Three-dimensional Poincaré supergravity and $\mathcal{N}$-extended supersymmetric BMS_3 algebra, Phys. Lett. B 792 (2019) 93, arXiv:1812.05065 [hep-th].
[81] R. Caroca, P. Concha, O. Fierro, E. Rodríguez, On the supersymmetric extension of asymptotic symmetries in three spacetime dimensions, Eur. Phys. J. C 80 (2020) 29, arXiv:1908.09150 [hep-th].
[82] E. Bergshoeff, J. Izquierdo, T. Ortín, L. Romano, Lie algebra expansions and actions for non-relativistic gravity, J. High Energy Phys. 08 (2019) 048, arXiv:1904.08304 [hep-th].

[83] J.A. de Azcárraga, D. Gútiez, J.M. Izquierdo, Extended $D = 3$ Bargmann supergravity from a Lie algebra expansion, Nucl. Phys. B 946 (2019) 114706, arXiv:1904.12786 [hep-th].
[84] L. Romano, Non-relativistic four dimensional p-brane supersymmetric theories and Lie algebra expansion, arXiv:1906.08220 [hep-th].
[85] O. Kasikci, N. Ozdemir, M. Ozkan, U. Zorba, Three-dimensional higher-order Schrödinger algebras and Lie algebra expansions, J. High Energy Phys. 04 (2020) 067, arXiv:2002.03558 [hep-th].
[86] G. Barnich, A. Gomberoff, H.A. Gonzalez, The flat limit of three dimensional asymptotically anti-de Sitter spacetimes, Phys. Rev. D 86 (2012) 024020, arXiv:1204.3288 [hep-th].
[87] O. Miskovic, R. Olea, D. Roy, Vacuum energy in asymptotically flat 2+1 gravity, Phys. Lett. B 767 (2017) 258, arXiv:1610.06101 [hep-th].