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Four dimensional topological supergravities from transgression field theory

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Abstract: In this work, we propose a four-dimensional gauged Wess-Zumino-Witten model, obtained as a dimensional reduction from a transgression field theory invariant under the $\mathcal{N}=1$ Poincaré supergroup. For this purpose, we consider that the two gauge connections on which the transgression action principle depends are given by linear and non-linear realizations of the gauge group respectively. The field content of the resulting four-dimensional theory is given by the gauge fields of the linear connection, in addition to a set of scalar and spinor multiplets in the same representation of the gauge supergroup, which in turn, correspond to the coordinates of the coset space between the gauge group and the five-dimensional Lorentz group. We then decompose the action in terms of four-dimensional quantities and derive the corresponding equations of motion. We extend our analysis to the non- and ultra- relativistic regimes.

Keywords: Chern-Simons Theories, Classical Theories of Gravity, Supergravity Models

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Contents

1 Introduction

In the last decades, several gravitational theories have been introduced as alternatives to General Relativity. The most general theory that can be formulated in arbitrary dimensions and fulfill the fundamental requirements of invariance under diffeomorphisms, lead to equations of motion of second degree in the metric tensor and preserve the conservation law of the energy-momentum tensor is known as the Lovelock theory [1, 2]. The Lovelock Lagrangian density is a sum over all the possible combinations between the Lorentz curvature depending on the spin connection, and the vielbein that codifies the metricity. In three and four dimensions, the Lovelock theory reproduces General Relativity with positive, negative, or zero cosmological constant depending on the values of its arbitrary constants [3–6]. A case of special interest is obtained when the constants of the sum in the Lovelock Lagrangian are fixed such that the theory presents the maximum number of degrees of freedom [7]. In that case, the three-dimensional Lovelock Lagrangian becomes proportional to the Chern–Simons (CS) three-form of the AdS group [8, 9], which has a topological origin and is, up to boundary terms, invariant under the gauge transformations induced by the AdS group. Such a feature allows us to formulate General Relativity as a topological gauge theory in three dimensions. However, the same does not occur in four dimensions, since CS forms exist only in odd dimensions. Moreover, the CS form of the AdS group does not lead to General Relativity in dimensions higher than three in any regime. With the purpose of formulating even-dimensional gravity theories, especially in four dimensions, A. H. Chamseddine proposed a topological gravity depending on the same variables than the Lovelock theory, in addition to a scalar multiplet in the same representation of the gauge group [10–12].

From a mathematical point of view, CS forms appear in the study of topological invariant densities that only exist in even dimension and, as a consequence of the Poincaré lemma, allow the existence odd-dimensional secondary forms that inherit their gauge invariance properties. These forms known as transgression forms, become CS forms as locally defined particular cases [13–18]. Thus, both transgression and CS forms are often used as Lagrangian densities for odd-dimensional gauge theories for gravity and supergravity. In contrast to CS theories, transgression field theories depend on two gauge connections and their Lagrangian densities are globally defined differential forms that, as well as the topological densities in which they originate, are fully gauge invariant. In addition to being useful in the construction of Lagrangian densities, transgression forms can naturally induce a dimensional reduction that allows to formulate even-dimensional gauge invariant theories without the need of breaking the gauge covariance. These are known gauged Wess–Zumino–Witten (gWZW) models, [19, 20] and, as well as the even-dimensional topological gravity proposed by Chamseddine, they include a scalar multiplet in the gauge group representation as part of the fundamental field content. In refs. [19–21], it was studied the relation between four-dimensional gWZW models and general relativity. Moreover, for an interesting study on the relation between gWZW models and the MacDowell–Mansouri gravity action, see ref. [22]. Furthermore, it was shown in [23, 24] that Chamseddine's topological gravities can be obtained as gWZW models of the Poincaré group in arbitrary even dimensions. These results were extended to the cases of the Maxwell algebra and the Poincaré superalgebra in three dimensions in ref. [25], obtaining fully gauge invariant $(1+1)$-dimensional theories for gravity and supergravity respectively.

The existence of abstract even-dimensional gWZW models motivates us to study the four-dimensional gauge invariant supergravity theory that emerges from five-dimensional Poincaré CS supergravity. Moreover, due to the well-known relation between the Poincaré group and the Galilei and Carroll groups [26], we aim the purpose of studying the non- and ultra- relativistic regimes of the resulting theory and thus to obtain the corresponding superalgebras and gWZW action principles.

It is important to clarify that CS supergravities, in general, do not fulfill the requirement of standard supergravity of having an equal number of bosonic and fermionic degrees of freedom, with some notable exceptions as the three-dimensional AdS supergravity or the five-dimensional AdS supergravity for $\mathcal{N}=3$ and $\mathcal{N}=5$. Consequently, the gWZW models that emerge from them will inherite this feature. For details on the differences on the formulations of standard and CS supergravity theories, see ref. [27].

This paper is organized as follows: In section 2, we consider a brief introduction to the SW-GN formalism and the gWZW model. In section 3, we study the non-linear realization of the five-dimensional Poincaré superalgebra. In section 4 and 5, we derive the four-dimensional gWZW action invariant under the aforementioned Poincaré supergroup, and study the dynamics of the resulting theory. In section 6, we consider the non-relativistic limit of the Poincaré superalgebra and derive the corresponding gWZW non-relativistic action principle. In section 7 we perform the same analysis for the ultra-relativistic limit of the theory. Section 8 contains our final conclusions.

2 Non-linear realizations and gWZW actions

In this section we briefly review the Stelle-West-Grignani-Nardelli (SW-GN) formalism and the gauged Wess-Zumino-Witten (gWZW) models.

2.1 The SW-GN formalism

The SW-GN formalism makes use of non-linear realizations of Lie groups in the construction of gauge invariant action principles [28–31]. Thus, the gauge symmetry of a physical theory can be extended from a stability group, to a higher-dimensional group that contains it as a subgroup. Let us consider a Lie group G with Lie algebra $\mathcal{G}$, and a subgroup $H \subset G$ with Lie algebra $\mathcal{H}$ as stability subgroup. We denote as $\{V_i\}_{i=1}^{\dim H}$ to the basis of $\mathcal{H}$ and as $\{T_l\}_{l=1}^{\dim G - \dim H}$ the to the set of generators of the remaining subspace. We assume that the basis can be chosen such that the generators T_l form a representation of the stability subgroup. Therefore, the Lie products between the vectors of the introduced basis satisfy $[V,T] \backsim T$, i.e. these products are linear combinations of T_l. An arbitrary group element g can be decomposed in terms the generators of the subgroup and the remaining subspace as

$$g = e^{\xi^l T_l} h \,, \tag{2.1}$$

where $h \in H$ is a group element defined by the action of the group on the zero-forms ξ^l which, in turn, play the role of coordinates that parametrize the coset space G/H. From eq. (2.1), it follows that the action of an arbitrary element $g_0 \in G$ on $e^{\xi^l T_l}$ can be also split as

$$g_0 e^{\xi^l T_l} = e^{\xi'^l T_l} h_1 \,. \tag{2.2}$$

Eq. (2.2) allows to obtain the non-linear functions $\xi' = \xi'(g,\xi)$ and $h_1 = h_1(g,\xi)$. By considering that the transformation law $\xi \to \xi'$ is described by the variation δ, and by choosing the group element g_0 such that $(g_0 - 1)$ is infinitesimal, eq. (2.2) leads to [32–34]

$$e^{-\xi^l T_l}(g_0 - 1)\, e^{\xi^l T_l} - e^{-\xi^l T_l} \delta e^{\xi^l T_l} = h_1 - 1 \,. \tag{2.3}$$

Since $g - 1$ is infinitesimal, $h_1 - 1$ is a vector of H.

Let us now consider the case in which $g_0 = h_0$ belongs to the stability subgroup. In this case, eq. (2.2) becomes

$$e^{\xi'^l T_l} = \left(h_0 e^{\xi^l T_l} h_0^{-1}\right) h_0 h_1^{-1} \,, \tag{2.4}$$

and since the Lie product $[V_i, T_l]$ is proportional to T_l, one gets $h_0 = h_1$ and the transformation law becomes linear:

$$e^{\xi'^l T_l} = h_0 e^{\xi^l T_l} h_0^{-1} \,. \tag{2.5}$$

On the other hand, if we consider $g_0 = e^{\xi_0^l T_l}$, eq. (2.2) becomes

$$e^{\xi'^l T_l} = e^{\xi_0^l T_l} e^{\xi^l T_l} h^{-1} \,, \tag{2.6}$$

which is a non-linear transformation law for ξ.

JHEP05(2024)248

Let us now consider a one-form gauge connection A taking values on $\mathcal{G}$ and an action principle $S = S[A]$ with gauge invariance under the transformations of the stability subgroup $\mathcal{H}$ but not under those along the generators of the coset space. Under the action of an arbitrary group element g, the gauge connection transforms as

$$A \longrightarrow A' = g^{-1}dg + g^{-1}Ag\,. \tag{2.7}$$

We split μ into its contributions belonging in h and the coset space as $A = a + \rho$, with $a = a^l T_l$ and $\rho = \rho_i V_i$. Moreover, we introduce a group element $z = \exp\left(\xi^l A_l\right)$ and define the non-linear gauge connection

$$A^z = z^{-1}\mathrm{d}z + z^{-1}Az\,. \tag{2.8}$$

The functional form of A^z is given by a large gauge transformation of A that non-linearly depends on the zero-forms ξ^l and their derivatives. However, in the SW-GN formalism A^z is interpreted as the fundamental field of a gauge theory and therefore, both A and ξ will change under the action of the gauge group. As before, we split the contributions to the non-linear connection as $A^z = v + p$ with

$$\begin{aligned} p &= p^l\left(\xi, \mathrm{d}\xi\right) T_l\,, \\ v &= v^i\left(\xi, \mathrm{d}\xi\right) V_i\,. \end{aligned} \tag{2.9}$$

It is possible to prove that, under the transformation δ generated by the action of the group, the transformation laws for p and v are given by

$$\begin{aligned} p &\longrightarrow p' = h_1^{-1} p h_1\,, \\ v &\longrightarrow v' = h_1^{-1} v h_1 + h_1^{-1}\mathrm{d}h_1\,, \end{aligned} \tag{2.10}$$

i.e., when acting with a group element belonging to the coset space, the non-linear one-forms p and v transform as a tensor and as a connection respectively. These transformations are linear but the group element is now a function of the parameters $h_1 = h_1\left(\xi_0, \xi\right)$. From the transformations laws in eq. (2.10), it follows that the non-linear gauge connection transforms in the same way that under the action of the stability subgroup and the coset space. Therefore, an action principle defined as a functional of A whose gauge symmetry is described by the stability subgroup, becomes invariant under the entire group G when A is replaced by A^z. The original non-invariance of the action principle is thus compensated by the transformation law of the gauge parameters ξ. For examples of the use of nonlinear realizations in gravity theories, see refs. [35, 36].

2.2 gWZW models

Let us consider two independent gauge connections A_1 and A_2 evaluated in the same gauge algebra. The transgression $(2n+1)$-form corresponding to both gauge connections is defined as

$$Q^{(2n+1)}_{A_2 \leftarrow A_1} = (n+1)\int_0^1 \langle\left(A_2 - A_1\right) F_t^n\rangle\,, \tag{2.11}$$

where $\langle\;\rangle$ denotes the symmetrized trace along the generators of the Lie algebra, and F_t is the gauge curvature associated to the homotopic gauge connection $A_t = A_1 + t\left(A_2 - A_1\right)$.

Transgression forms are globally defined and fully invariant under the transformations of the gauge group. CS forms emerge as particular cases of transgression forms, by locally setting one of the gauge connections as vanishing. Thus, the CS form corresponding to a gauge connection A is locally defined as

$$Q^{(2n+1)}_{A\leftarrow 0} = (n+1)\int_0^1 \langle A F_t^n \rangle \,, \tag{2.12}$$

where the homotopic gauge connection takes form $A_t = tA$. Furthermore, by applying the Cartan homotopy formula, it is possible to prove that a general transgression form can be written in terms of two CS forms and a total derivative, as follows [14, 15, 37]

$$Q^{(2n+1)}_{A_2\leftarrow A_1} = Q^{(2n+1)}_{A_2\leftarrow A_0} - Q^{(2n+1)}_{A_1\leftarrow A_0} - Q^{(2n)}_{A_2\leftarrow A_1\leftarrow A_0} \,. \tag{2.13}$$

The $2n$-form inside the exterior derivative is explicitly given as the following integral:

$$Q^{(2n)}_{A_2\leftarrow A_1\leftarrow A_0} = n\,(n+1)\int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s \left\langle (A_2 - A_1)\,(A_1 - A_0)\, F_{st}^{n-1} \right\rangle \,, \tag{2.14}$$

where F_{st} is the gauge curvature associated to the homotopic gauge field

$$A_{st} = A_0 + t\,(A_1 - A_0) + s\,(A_2 - A_1) \,, \tag{2.15}$$

which depends on two parameters t and s taking values between 0 and 1. For details on the use of the Cartan homotopy formula and the homotopy operator in this context, see refs. [14, 15, 17, 38].

Let us now consider two gauge connections A and A^z, related by the gauge transformation $A^z = z^1\,(\mathrm{d} + A)\,z$, where $z = \exp(\xi)$ is an element of the gauge group and ξ a zero-form mutiplet in the same representation of the Lie algebra. From eq. (2.19), it follows that the transgression form associated to A and A^z can be written in terms of the difference between their corresponding CS forms. Let us now introduce a homotopic gauge field $A_t = tA$ which takes values between 0 and A, as the parameter t takes values between 0 and 1. The transformed connection, obtained from A_t, denoted by $(A_t)^z$, and its corresponding gauge curvature $(F_t)^z$ are given by[1]

$$\begin{aligned} (A_t)^z &= z^{-1} t A z + z^{-1}\mathrm{d}z \,, \\ (F_t)^z &= z^{-1} F_t z = z^{-1}\left(tF + \left(t^2 - t\right) A^2\right) z \,. \end{aligned} \tag{2.16}$$

These homotopic quantities verify

$$\begin{aligned} (A_0)^z &= z^{-1}\nu z = z^{-1}\mathrm{d}z, \qquad (A_1)^z = A^z \,, \\ (F_0)^z &= 0, \qquad (F_1)^z = F^z \,. \end{aligned} \tag{2.17}$$

By applying the Cartan homotopy formula once again, it is possible to prove that the CS forms corresponding to the pure gauge connection $z^{-1}\mathrm{d}z$ and the transformed gauge connections A^z are related by the following equation

$$Q^{(2n+1)}_{A^z\leftarrow 0} - Q^{(2n+1)}_{z^{-1}dz\leftarrow 0} = (k_{01}\mathrm{d} + \mathrm{d}k_{01})\, Q^{(2n+1)}_{(A_t)^z\leftarrow 0} \,, \tag{2.18}$$

[1] Note that in general $(A_t)^z \neq (A^z)_t$.

JHEP05(2024)248

with $k_{01} = \int_0^1 \ell_t$, where ℓ_t is the homotopy operator defined by the following action on A_t and F_t:

$$\ell_t A_t = 0, \qquad \ell_t F_t = \mathrm{d}t \frac{\partial}{\partial t} A_t \,. \tag{2.19}$$

By directly applying eq. (2.19), one finds that the first term in the right side of (2.18) is given by

$$k_{01} \mathrm{d} Q^{(2n+1)}_{(A_t)^z \leftarrow 0} = Q^{(2n+1)}_{A \leftarrow 0} \,, \tag{2.20}$$

so that, eq. (2.18) allows to write the difference between two CS forms related by means of a gauge connection as

$$Q^{(2n+1)}_{A^z \leftarrow 0} - Q^{(2n+1)}_{A \leftarrow 0} = Q^{(2n+1)}_{z^{-1}\mathrm{d}z \leftarrow 0} + \mathrm{d}\alpha_{2n}\left(A, z\right) \,, \tag{2.21}$$

where we introduce

$$\alpha_{2n}\left(A, z\right) = k_{01} Q^{(2n+1)}_{(A_t)^z \leftarrow 0} \,. \tag{2.22}$$

Notice that the first term in the r.h.s. of eq. (2.21) is the CS form corresponding to the pure gauge connection $z^{-1}\mathrm{d}z$, which is explicitly given by

$$Q^{(2n+1)}_{z^{-1}\mathrm{d}z \leftarrow 0} = (-1)^n \frac{n!\,(n+1)!}{(2n+1)!} \left\langle \left(z^{-1}\mathrm{d}z\right)^{2n+1} \right\rangle \,. \tag{2.23}$$

Thus, by virtue of eq. (2.13), it is possible to write down the transgression form corresponding to A and A^z in terms of the pure gauge connection and a total derivative

$$Q^{(2n+1)}_{A^z \leftarrow A} = Q^{(2n+1)}_{z^{-1}\mathrm{d}z \leftarrow 0} + \mathrm{d}\left(\alpha_{2n}\left(A, z\right) - Q^{(2n)}_{A^z \leftarrow A \leftarrow 0}\right) \,. \tag{2.24}$$

Given a gauge group, the so-called gWZW action is defined as the boundary action that appears from the transgression action in accordance with the Stoke's theorem [19, 20]

$$\begin{aligned} S_{\mathrm{gWZW}}\left[A\right] &= \kappa \int_M Q^{(2n+1)}_{A^z \leftarrow A} \\ &= \kappa \int_{\partial M} \alpha_{2n}\left(A, z\right) - Q^{(2n)}_{A^z \leftarrow A \leftarrow 0} \,. \end{aligned} \tag{2.25}$$

Since the transgression Lagrangian is odd-dimensional, the gWZW action principle is always even-dimensional. As it happens in the SW-GN formalism, the zero-forms ξ are not longer interpreted as the parameters of a symmetry transformation but as physical fields with a topological origin. However, in contrast with the gauge invariant action principles that are obtained in the SW-NG formalism, gWZW action principles are not exclusively functionals of the non-linear gauge fields, but of the linear ones and the zero-form multiplets.

3 $\mathcal{N} = 1$ Poincaré supergravity

3.1 Chern–Simons supergravity

The construction of a four-dimensional gWZW model requires the five-dimensional CS action as starting point. We first consider the $\mathcal{N} = 1$ supersymmetric extension of the five-dimensional Poincaré algebra $\mathfrak{u}\left(4|1\right)$, which is spanned by the set of generators

$\left\{J_{AB}, P_A, K, Q^\alpha, \bar{Q}_\alpha\right\}$ where $\bar{Q}_\alpha$ and Q^α are independent Dirac spinors. Capital latin letters denote five-dimensional Lorentz indices taking values as $A = 0, \dots 4$, while Greek letters denote spinor indices taking values as $\alpha = 1, \dots 4$. In the chosen basis, the (anti)conmutation relations between the introduced generators are given by [9, 15, 17]

$$\begin{aligned}
[J_{AB}, P_C] &= \eta_{BC} P_A - \eta_{AC} P_B \,, \\
[J_{AB}, J_{CD}] &= \eta_{BC} J_{AD} + \eta_{AD} J_{BC} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC} \,, \\
[J_{AB}, Q^\alpha] &= -\frac{1}{2} \left(\Gamma_{AB}\right)^\alpha{}_\beta Q^\beta \,, \\
\left[J_{AB}, \bar{Q}_\alpha\right] &= \frac{1}{2} \left(\Gamma_{AB}\right)^\beta{}_\alpha \bar{Q}_\beta \,, \\
\left\{Q^\alpha, \bar{Q}_\beta\right\} &= 2 \left(\Gamma^A\right)^\alpha{}_\beta P_A - 4i \delta^\alpha_\beta K \,,
\end{aligned} \tag{3.1}$$

where the metric signature is chosen as $\eta^{AB} = \operatorname{diag}(-, +, +, +, +)$. This superalgebra allows an inner invariant rank-3 product with the following components:

$$\begin{aligned}
\langle K J_{AB} J_{CD} \rangle &= -\frac{i}{4} \left(\eta_{AC}\eta_{BD} - \eta_{BC}\eta_{AD}\right) \,, \\
\langle J_{AB} J_{CD} P_E \rangle &= \frac{1}{2} \epsilon_{ABCDE} \,, \\
\left\langle Q^\alpha J_{AB} \bar{Q}_\beta \right\rangle &= -\left(\Gamma_{AB}\right)^\alpha{}_\beta \,.
\end{aligned} \tag{3.2}$$

We gauge the algebra by considering a one-form gauge connection A with non-vanishing gauge curvature $F = \mathrm{d}A + \frac{1}{2}[A, A]$, to whose components we denote

$$\begin{aligned}
A &= h^A P_A + \frac{1}{2} \omega^{AB} J_{AB} + bK + \bar{\psi}_\alpha Q^\alpha - \bar{Q}_\alpha \psi^\alpha, \\
F &= \mathcal{T}^A P_A + \frac{1}{2} \mathcal{R}^{AB} J_{AB} + F_b K + \bar{\mathcal{F}}_\alpha Q^\alpha - \bar{Q}_\alpha \mathcal{F}^\alpha.
\end{aligned} \tag{3.3}$$

The components of the gauge curvature are given explicitly by

$$\begin{aligned}
\mathcal{T}^A &= \mathrm{d}h^A + \omega^A{}_C h^C - 2\bar{\psi}_\alpha \left(\Gamma^A\right)^\alpha{}_\beta \psi^\alpha \,, \\
\mathcal{R}^{AB} &= \mathrm{d}\omega^{AB} + \omega^A{}_C \omega^{CB} \,, \\
F_b &= \mathrm{d}b + 4i \delta^\alpha_\beta \bar{\psi}_\alpha \psi^\alpha \,, \\
\bar{\mathcal{F}}_\alpha &= \mathcal{D}\bar{\psi}_\alpha \equiv \mathrm{d}\bar{\psi}_\alpha - \frac{1}{4} \omega^{AB} \bar{\psi}_\beta \left(\Gamma_{AB}\right)^\beta{}_\alpha \,, \\
\mathcal{F}^\alpha &= \mathcal{D}\psi \equiv \mathrm{d}\psi^\alpha + \frac{1}{4} \omega^{AB} \left(\Gamma_{AB}\right)^\alpha{}_\beta \psi^\beta \,,
\end{aligned} \tag{3.4}$$

where $\mathcal{D}$ denotes the covariant derivative defined with respect to the five-dimensional spin connection ω^{AB}. By using the subspace separation procedure (see refs. [15, 17]), it is possible to write down the five-dimensional CS Lagrangian in a convenient way:

$$\mathcal{L}_{\mathrm{CS}}(A) = \kappa \left(\frac{1}{4} \epsilon_{ABCDE} \mathcal{R}^{AB} \mathcal{R}^{CD} h^E + \frac{i}{4} \mathcal{R}^{AB} \mathcal{R}_{AB} b - \left(\bar{\psi} \mathcal{R}^{AB} \Gamma_{AB} \mathcal{D}\psi + \mathcal{D}\bar{\psi} \mathcal{R}^{AB} \Gamma_{AB} \psi \right) \right) , \tag{3.5}$$

where κ is a constant.

3.2 Non-linear realization

In order to perform the dimensional reduction, it is necessary to introduce a non-linear realization of the gauge supergroup. We therefore consider a second gauge connection A^z related with A by means of the following large gauge transformation

$$A^z = z^{-1}\left(\mathrm{d} + A\right) z. \tag{3.6}$$

Here, z is an element of the gauge group, specifically in the coset space $\mathcal{G}_5/SO\left(4,1\right)$. Taking in account the following decomposition of the Poincaré superalgebra

$$\begin{aligned} &L_0 = \left\{J_{AB}\right\}, \quad L_1 = \left\{J_{AB}, P_A\right\}, \quad L_3 = \left\{J_{AB}, P_A, K\right\}, \\ &L_4 = \left\{J_{AB}, P_A, K, \bar{Q}_\alpha\right\}, \quad L_5 = \left\{J_{AB}, P_A, K, \bar{Q}_\alpha\right\}, \end{aligned} \tag{3.7}$$

we can express A^z by using the following gauge group element

$$z = z_{\bar{\chi}} z_\chi z_\varphi z_\phi = e^{-\bar{\chi}_\alpha Q^\alpha} e^{\bar{Q}_\beta \chi^\beta} e^{-\varphi K} e^{-\phi^A P_A}, \tag{3.8}$$

where φ and ϕ^A are zero-forms, and where χ and $\bar{\chi}_\alpha$ are Dirac spinors zero-forms. In order to explicitly write down for A^z in terms of the components of A and the parameters of the gauge transformation, we use the following identities [39]

$$\begin{aligned} e^X \delta e^{-X} &= \frac{\left(1 - e^x\right)}{X} \wedge \delta X, \\ e^X Y e^{-X} &= e^X \wedge Y, \end{aligned} \tag{3.9}$$

with the notation $X \wedge Y = [X, Y]$ and where δ is any variation. By directly and successively applying in the (anti)commutation relations of the gauge algebra into eq. (3.6), we obtain the following transformed gauge field

$$A^z = V + W + \tilde{B} + \bar{\Psi} - \Psi, \tag{3.10}$$

where each component is given by

$$\begin{aligned} V &= V^A P_A = \left(h^A - \mathcal{D}\phi^A + 2\bar{\psi}\Gamma^A\chi - 2\mathcal{D}\bar{\chi}\Gamma^A\chi - 2\bar{\chi}\Gamma^A\psi\right) P_A, \\ W &= \frac{1}{2} W^{AB} J_{AB} = \frac{1}{2}\omega^{AB} J_{AB}, \\ \tilde{B} &= BK = \left\{b - \mathrm{d}\varphi + 4i\left(\left(\mathcal{D}\bar{\chi}\right)\chi - \bar{\psi}\chi + \bar{\chi}\psi\right)\right\} K, \\ \bar{\Psi} &= \bar{\Psi} Q = \left(\bar{\psi} - \mathcal{D}\bar{\chi}\right) Q, \\ \Psi &= \bar{Q}\Psi = \bar{Q}\left(\psi - \mathcal{D}\chi\right). \end{aligned} \tag{3.11}$$

The non-linear realization of the gauge algebra allows the construction of invariant action principles. In fact, the five-dimensional standard supergravity theory, whose Lagrangian functional includes the Einstein–Hilbert and Rarita–Schwinger terms becomes invariant under the Poincaré superalgebra when one identifies the non-linear field V^A as the fünfbein field associated to the metric tensor of the supergravity theory, and Ψ^α and $\bar{\Psi}_\alpha$ as the spin 3/2 fields.

4 Four-dimensional Poincaré supergravity

In order to write down the transgression form depending on both gauge connections A and A^z, let us recall (3.8). By inspection of the (anti)commutation relations and invariant tensors of the gauge algebra, it follows that, in this case, the pure gauge connection $z^{-1}\mathrm{d}z$ has no components along the Lorentz rotation generators. Therefore, the pure gauge contribution to the transgression form, vanishes for any gauge parameter lying in $\mathfrak{u}\,(4|1)\,/\mathfrak{so}\,(4,1)$

$$Q^{(5)}_{z^{-1}\mathrm{d}z\leftarrow 0}=0\,. \tag{4.1}$$

As a consequence, the transgression form $Q^{(5)}_{A^z\leftarrow A}$ is always exact and, according eq. (2.24), can be written as

$$Q^{(5)}_{A^z\leftarrow A}=\mathrm{d}\left(\alpha_4\,(A,z)-Q^{(4)}_{A^z\leftarrow A\leftarrow 0}\right)\,. \tag{4.2}$$

To find an explicit expression of this transgression, let us first consider eq. (2.13) with the following choice of gauge connections:

$$Q^{(5)}_{A^z\leftarrow A}=Q^{(5)}_{A^z\leftarrow\omega}-Q^{(5)}_{A\leftarrow\omega}-\mathrm{d}Q^{(4)}_{A^z\leftarrow A\leftarrow\omega}\,. \tag{4.3}$$

Notice that we now choose the intermediate connection as $\bar{A}=\omega$. A direct calculation shows that the difference between both transgression forms in the r.h.s. of eq. (4.3) is given by

$$\begin{aligned}Q^{(5)}_{A^z\leftarrow\bar{A}}-Q^{(5)}_{A\leftarrow\bar{A}}=\mathrm{d}&\left(\frac{3}{8}\epsilon_{ABCDE}\mathcal{R}^{AB}\mathcal{R}^{CD}\phi^E+\frac{3i}{8}\mathcal{R}^{AB}\mathcal{R}_{AB}\varphi+3i\left\langle\mathcal{D}\bar{\chi}\mathcal{R}\psi-\bar{\psi}\mathcal{R}\mathcal{D}\chi+\mathcal{D}\bar{\chi}\mathcal{R}\mathcal{D}\chi\right\rangle\right)\\&-\frac{3}{4}\epsilon_{ABCDE}\mathcal{R}^{AB}\mathcal{R}^{CD}\left(\bar{\psi}\Gamma^E\chi-\mathcal{D}\bar{\chi}\Gamma^E\chi-\bar{\chi}\Gamma^E\psi\right)+\frac{3}{2\ell}\mathcal{R}^{AB}\mathcal{R}_{AB}\left((\mathcal{D}\bar{\chi})\,\chi-\bar{\psi}\chi+\bar{\chi}\psi\right)\\&-3\left(\mathcal{D}^2\bar{\chi}\mathcal{R}^{AB}\Gamma_{AB}\psi+\bar{\psi}\mathcal{R}^{AB}\Gamma_{AB}\mathcal{D}^2\chi-\mathcal{D}\bar{\chi}\mathcal{R}^{AB}\Gamma_{AB}\mathcal{D}^2\chi\right)\,,\end{aligned} \tag{4.4}$$

where letters carrying no index inside the trace are vectors of the Lie superalgebra, i.e., $\mathcal{R}=\frac{1}{2}\mathcal{R}^{AB}J_{AB}$, $\psi=\bar{Q}_\alpha\psi^\alpha$, $\bar{\psi}=\bar{\psi}_\alpha Q^\alpha$, $\chi=\bar{Q}_\alpha\chi^\alpha$ and $\bar{\chi}=\bar{\chi}_\alpha Q^\alpha$. By using the Bianchi identities, we have that the second derivatives of the fermion zero-forms can be written in terms of the Lorentz curvature as

$$\mathcal{D}^2\bar{\chi}_\alpha=-\frac{1}{4}\bar{\chi}_\beta\mathcal{R}^{AB}\left(\Gamma_{AB}\right)^\beta{}_\alpha\,,\qquad\qquad\mathcal{D}^2\chi^\alpha=\frac{1}{4}\mathcal{R}^{AB}\left(\Gamma_{AB}\right)^\alpha{}_\beta\,\chi^\beta\,. \tag{4.5}$$

Then, by integrating by parts and using the properties of the gamma matrices in five dimensions, we find that the difference between both transgression forms is given by the following total derivative

$$Q^{(5)}_{A^z\leftarrow\bar{A}}-Q^{(5)}_{A\leftarrow\bar{A}}=\mathrm{d}\left[\frac{3}{8}\epsilon_{ABCDE}\mathcal{R}^{AB}\mathcal{R}^{CD}\phi^E+\frac{3i}{8}\mathcal{R}^{AB}\mathcal{R}_{AB}\varphi+3i\left\langle\mathcal{D}\bar{\chi}\mathcal{R}\psi-\bar{\psi}\mathcal{R}\mathcal{D}\chi+\mathcal{D}\bar{\chi}\mathcal{R}\mathcal{D}\chi\right\rangle\right]. \tag{4.6}$$

On the other hand, the the boundary term in the r.h.s. of eq. (4.3) can be obtained from eq. (2.14) by setting $n=2$, $A_2=A^z$, $A_1=A$ and $\bar{A}=\omega$. Consequently, the homotopic gauge field becomes $A_{st}=\omega+s\,(A^z-A)+t\,(A-\omega)$. A direct integration leads to

$$Q^{(4)}_{A^z\leftarrow A\leftarrow\bar{A}}=-3i\left\langle\mathcal{D}\bar{\chi}\mathcal{R}\psi-\bar{\psi}\mathcal{R}\mathcal{D}\chi\right\rangle\,. \tag{4.7}$$

JHEP05(2024)248

Then, by plugging in eqs. (4.6) and (4.7) into (4.3), we finally obtain an explicit expression for the transgression form in terms of a total derivative

$$
\begin{aligned}
Q^{(5)}_{A^z\leftarrow A} = \mathrm{d}\Big\{ & \frac{3}{8}\epsilon_{ABCDE}\mathcal{R}^{AB}\mathcal{R}^{CD}\phi^E + \frac{3i}{8}\mathcal{R}^{AB}\mathcal{R}_{AB}\varphi \\
& +3\left[\mathcal{D}\bar{\chi}\mathcal{R}^{AB}\Gamma_{AB}\left(\psi + \frac{1}{4}\mathcal{D}\chi\right) - \left(\bar{\psi} - \frac{1}{4}\mathcal{D}\bar{\chi}\right)\mathcal{R}^{AB}\Gamma_{AB}\mathcal{D}\chi\right]\Big\}\,. \qquad (4.8)
\end{aligned}
$$

Therefore, the four-dimensional induced action is given by

$$
\begin{aligned}
S = \kappa\int\Big[& \epsilon_{ABCDE}\mathcal{R}^{AB}\mathcal{R}^{CD}\phi^E + i\mathcal{R}^{AB}\mathcal{R}_{AB}\varphi \\
& -8\left(\bar{\chi}\mathcal{R}^{AB}\Gamma_{AB}\mathcal{D}\psi + \mathcal{D}\bar{\psi}\mathcal{R}^{AB}\Gamma_{AB}\chi - \frac{1}{2}\mathcal{D}\bar{\chi}\mathcal{R}^{AB}\Gamma_{AB}\mathcal{D}\chi\right)\Big]\,. \qquad (4.9)
\end{aligned}
$$

This action in analogue to the one found in ref. [25] for $(1+1)$-dimensional supergravity. It is invariant under the transformations of the five-dimensional Poincaré supergroup and it can be interpreted as a supersymmetric extension of the topological gravity introduced in ref. [10], and alternatively found as a gWZW action in ref. [23] for the bosonic case.

4.1 Decomposition of the action

The obtained gWZW action from eq. (4.9) is four-dimensional. However, it is a functional of the original five-dimensional field content of the transgression field theory. When gauging Poincaré or AdS supergroups, the gauge field associated to the translation operator is usually identified as the vielbein of the corresponding supergravity theory, and therefore it is considered that it carries the information about the metric in the resulting field equations. Notice that, at this point, we have not yet introduced a notion of metricity in the supergravity theory. Moreover, the field h^A has been removed from the functional in the dimensional reduction process and it is not longer present in the action principle in eq. (4.9). This is a common feature of gWZW models originated in the gauging of space-time symmetries, and allows us to identify the some components of the five-dimensional spin connection as vierbein in a four-dimensional supergravity theory. Thus, the original gauge invariance under the five-dimensional Lorentz group is now interpreted as invariance under the four-dimensional de Sitter group. We therefore decompose the index $A=(a,4)$ with $a=0,1,2,3$, and rename $\omega^{a4}=-\omega^{4a}=e^a$ as vierbein one-form. The five-dimensional Lorentz curvature is also decomposed, as follows:

$$
\begin{aligned}
\mathcal{R}^{ab} &= \mathrm{d}\omega^{ab} + \omega^{a}{}_{c}\omega^{cb} + \omega^{a}{}_{4}\omega^{4b} = R^{ab} - e^a e^b\,, \\
\mathcal{R}^{a4} &= \mathrm{D}e^a = T^a\,. \qquad (4.10)
\end{aligned}
$$

Consequently, we split the action principle into its bosonic and fermionic sectors as $S=S_{\mathrm{B}}+S_{\mathrm{F}}$, being S_{F} the contribution depending on spinor fields in the r.h.s. of eq. (4.9). In terms of the previous decomposition, the bosonic sector of the action is given by

$$
\begin{aligned}
S_{\mathrm{B}} = \kappa\int\Big[& -4\left(\epsilon_{abcd}R^{ab}e^c - \frac{1}{3}\epsilon_{abcd}e^a e^b e^c\right)\mathrm{D}\phi^d \\
& +\epsilon_{abcd}\left[R^{ab}R^{cd} - 2R^{ab}e^c e^d + e^a e^b e^c e^d\right]\phi^4 + iR^{ab}R_{ab}\varphi + 2iT^a e_a \mathrm{d}\varphi\Big]\,. \qquad (4.11)
\end{aligned}
$$

In the same way, the fermionic sector of the action is given in terms of the four-dimensional quantities as follows

$$S_{\mathrm{F}} = \kappa \int -8 \left[\bar{\chi} \left(R^{ab} - e^a e^b \right) \Gamma_{ab} \mathcal{D}\psi + 2\bar{\chi} T^a \Gamma_a \Gamma \mathcal{D}\psi + \mathcal{D}\bar{\psi} \left(R^{ab} - e^a e^b \right) \Gamma_{ab} \chi \right.$$
$$\left. - \frac{1}{2} \mathcal{D}\bar{\chi} \left(R^{ab} - e^a e^b \right) \Gamma_{ab} \mathcal{D}\chi - \mathcal{D}\bar{\chi} T^a \Gamma_a \Gamma \mathcal{D}\chi \right], \tag{4.12}$$

where the five-dimensional Lorentz covariant derivatives are given in terms of the four-dimensional ones according to[2]

$$\mathcal{D}\bar{\psi} = \mathrm{D}\bar{\psi} - \frac{1}{2} e^a \bar{\psi} \Gamma_a \Gamma, \qquad \mathcal{D}\psi = \mathrm{D}\psi + \frac{1}{2} e^a \Gamma_a \Gamma \psi \,. \tag{4.13}$$

5 Dynamics

From now on, we will denote as $\mathcal{L}_{\mathrm{G}}$ to the bosonic Lagrangian four-form, and identify to the fermionic contribution as a matter Lagrangian, i.e.,

$$\mathcal{L}_{\mathrm{G}} = \kappa \left[\epsilon_{ABCDE} \mathcal{R}^{AB} \mathcal{R}^{CD} \phi^E + i \mathcal{R}^{AB} \mathcal{R}_{AB} \varphi \right],$$
$$\mathcal{L}_{\mathrm{M}} = -8\kappa \left[\bar{\chi} \mathcal{R}^{AB} \Gamma_{AB} \mathcal{D}\psi + \mathcal{D}\bar{\psi} \mathcal{R}^{AB} \Gamma_{AB} \chi - \frac{1}{2} \mathcal{D}\bar{\chi} \mathcal{R}^{AB} \Gamma_{AB} \mathcal{D}\chi \right]. \tag{5.1}$$

Although we hold the writing in terms of the five-dimensional indices for convenience, it is important to recall that they describe a four-dimensional theory with ω^{AB} packing the spin connection and vielbein forms, while $\mathcal{R}^{AB}$ contains the Lorentz curvature and torsion. In these terms, we introduce a generalized spin form Σ_{AB}, such that the variation of $\mathcal{L}_{\mathrm{M}}$ with respect to ω^{AB} is given by

$$\delta_\omega \mathcal{L}_{\mathrm{M}} = -k \delta\omega^{AB} * \Sigma_{AB} \,, \tag{5.2}$$

with $*$ the Hodge dual operator and k a dimensional constant. The components of the generalized spin form are split into the four dimensional spin form and energy-momentum forms, as follows

$$\Sigma_{a4} = \mathcal{T}_a, \qquad \Sigma_{AB} = \sigma_{ab} \,. \tag{5.3}$$

Therefore, the field equations can be written as

$$\delta_\omega \mathcal{L}_{\mathrm{G}} - k \delta\omega^{AB} * \Sigma_{AB} = 0 \,. \tag{5.4}$$

The variation of $\mathcal{L}_{\mathrm{G}}$ with respect to ω^{AB} is given by

$$\delta_\omega \mathcal{L}_{\mathrm{G}} = 2\kappa \delta\omega^{AB} \left[\epsilon_{ABCDE} \mathcal{R}^{CD} \mathcal{D}\phi^E + i \mathcal{R}_{AB} \mathrm{d}\varphi \right]. \tag{5.5}$$

On the other hand, the field variation of the matter Lagrangian is given by

$$\delta_\omega \mathcal{L}_{\mathrm{M}} = -8\kappa \left[-\mathcal{D}\bar{\chi} \delta\omega^{AB} \Gamma_{AB} \mathcal{D}\psi + \bar{\chi} \delta\omega^{AB} \Gamma_{AB} \mathcal{D}^2 \psi + \frac{1}{4} \bar{\chi} \mathcal{R}^{AB} \delta\omega^{CD} \Gamma_{AB} \Gamma_{CD} \psi \right.$$
$$- \frac{1}{4} \delta\omega^{CD} \bar{\psi} \mathcal{R}^{AB} \Gamma_{CD} \Gamma_{AB} \chi - \mathcal{D}^2 \bar{\psi} \delta\omega^{AB} \Gamma_{AB} \chi + \mathcal{D}\bar{\psi} \delta\omega^{AB} \Gamma_{AB} \mathcal{D}\chi$$
$$+ \frac{1}{8} \delta\omega^{CD} \bar{\chi} \mathcal{R}^{AB} \Gamma_{CD} \Gamma_{AB} \mathcal{D}\chi - \frac{1}{8} \mathcal{D}\bar{\chi} \mathcal{R}^{AB} \delta\omega^{CD} \Gamma_{AB} \Gamma_{CD} \chi - \frac{1}{2} \mathcal{D}^2 \bar{\chi} \delta\omega^{AB} \Gamma_{AB} \mathcal{D}\chi$$
$$- \frac{1}{2} \mathcal{D}\bar{\chi} \delta\omega^{AB} \Gamma_{AB} \mathcal{D}^2 \chi \,, \tag{5.6}$$

[2]From now on, we denote $\Gamma^5 \equiv \Gamma$.

where we have used the identities

$$\delta\mathcal{D}\bar{\psi} = -\frac{1}{4}\delta\omega^{AB}\bar{\psi}\Gamma_{AB}, \quad \delta\mathcal{D}\psi = \frac{1}{4}\delta\omega^{AB}\Gamma_{AB}\psi\,,$$
$$\delta\mathcal{D}\bar{\chi} = -\frac{1}{4}\delta\omega^{AB}\bar{\chi}\Gamma_{AB}, \quad \delta\mathcal{D}\chi = \frac{1}{4}\delta\omega^{AB}\Gamma_{AB}\chi\,. \tag{5.7}$$

By integrating by parts and plugging in the Bianchi identities

$$\mathcal{D}^2\bar{\chi} = -\frac{1}{4}\mathcal{R}^{AB}\bar{\chi}\Gamma_{AB}, \quad \mathcal{D}^2\chi = \frac{1}{4}\mathcal{R}^{AB}\Gamma_{AB}\chi\,,$$
$$\mathcal{D}^2\bar{\psi} = -\frac{1}{4}\mathcal{R}^{AB}\bar{\psi}\Gamma_{AB}, \quad \mathcal{D}^2\psi = \frac{1}{4}\mathcal{R}^{AB}\Gamma_{AB}\psi\,, \tag{5.8}$$

we obtain

$$\begin{aligned}\delta_\omega\mathcal{L}_{\mathrm{M}} = {}& -8\kappa\delta\omega^{AB}\left[\mathcal{D}\bar{\chi}\Gamma_{AB}\mathcal{D}\psi + \mathcal{D}\bar{\psi}\Gamma_{AB}\mathcal{D}\chi + \mathcal{R}_{AB}\left(\bar{\psi}\chi - \bar{\chi}\psi - \frac{1}{2}\mathrm{d}\left(\bar{\chi}\chi\right)\right)\right.\\ &\left.+\frac{1}{2}\epsilon_{ABCDE}\mathcal{R}^{CD}\left(\bar{\psi}\Gamma^E\chi - \bar{\chi}\Gamma^E\psi - \frac{1}{2}\mathcal{D}\left(\bar{\chi}\Gamma^E\chi\right)\right)\right]\,.\end{aligned} \tag{5.9}$$

Finally we have an expression for the dual spin form

$$\begin{aligned}*\Sigma_{AB} = {}& \frac{8\kappa}{k}\left[\mathcal{D}\bar{\chi}\Gamma_{AB}\mathcal{D}\psi + \mathcal{D}\bar{\psi}\Gamma_{AB}\mathcal{D}\chi + \mathcal{R}_{AB}\left(\bar{\psi}\chi - \bar{\chi}\psi - \frac{1}{2}\mathrm{d}\left(\bar{\chi}\chi\right)\right)\right.\\ &\left.+\frac{1}{2}\epsilon_{ABCDE}\mathcal{R}^{CD}\left(\bar{\psi}\Gamma^E\chi - \bar{\chi}\Gamma^E\psi - \frac{1}{2}\mathcal{D}\left(\bar{\chi}\Gamma^E\chi\right)\right)\right]\,.\end{aligned} \tag{5.10}$$

The field equations coming from the variation of the action with respect to ω^{AB} are therefore given by $\varepsilon_{AB} = 0$ with

$$\begin{aligned}\varepsilon_{AB} = {}& \epsilon_{ABCDE}\mathcal{R}^{CD}\mathcal{D}\phi^E + i\mathcal{R}_{AB}d\varphi - 4\left[\mathcal{D}\bar{\chi}\Gamma_{AB}\mathcal{D}\psi + \mathcal{D}\bar{\psi}\Gamma_{AB}\mathcal{D}\chi\right.\\ &\left.+\mathcal{R}_{AB}\left(\bar{\psi}\chi - \bar{\chi}\psi - \frac{1}{2}\mathrm{d}\left(\bar{\chi}\chi\right)\right) + \frac{1}{2}\epsilon_{ABCDE}\mathcal{R}^{CD}\left(\bar{\psi}\Gamma^E\chi - \bar{\chi}\Gamma^E\psi - \frac{1}{2}\mathcal{D}\left(\bar{\chi}\Gamma^E\chi\right)\right)\right]\,,\end{aligned} \tag{5.11}$$

or, equivalently in components, the field equations related with the independent variations δe^a and $\delta\omega^{ab}$ are given by

$$\begin{aligned}\varepsilon_{a4} = {}& -\epsilon_{abcd}\mathcal{R}^{bc}\left(D\phi^d + e^d\phi^4\right) + iT_a\mathrm{d}\varphi\\ &-4\left[\mathcal{D}\bar{\chi}\Gamma_a\Gamma\mathcal{D}\psi + \mathcal{D}\bar{\psi}\Gamma_a\Gamma\mathcal{D}\chi + T_a\left(\bar{\psi}\chi - \bar{\chi}\psi - \frac{1}{2}\mathrm{d}\left(\bar{\chi}\chi\right)\right)\right.\\ &\left.-\frac{1}{2}\epsilon_{abcd}\mathcal{R}^{bc}\left(\bar{\psi}\Gamma^d\chi - \bar{\chi}\Gamma^d\psi - \frac{1}{2}\mathrm{D}\left(\bar{\chi}\Gamma^d\chi\right) - \frac{1}{2}e^d\bar{\chi}\Gamma^4\chi\right)\right]\,,\end{aligned} \tag{5.12}$$

$$\begin{aligned}\varepsilon_{ab} = {}& \epsilon_{abcd4}\mathcal{R}^{cd}\left(\mathrm{d}\phi^4 - e^b\phi_b\right) - 2\epsilon_{abcd}T^c\left(\mathrm{D}\phi^d + e^d\phi^4\right) + i\mathcal{R}_{ab}\mathrm{d}\varphi\\ &-4\left[\mathcal{D}\bar{\chi}\Gamma_{ab}\mathcal{D}\psi + \mathcal{D}\bar{\psi}\Gamma_{ab}\mathcal{D}\chi + \mathcal{R}_{ab}\left(\bar{\psi}\chi - \bar{\chi}\psi - \frac{1}{2}\mathrm{d}\left(\bar{\chi}\chi\right)\right)\right.\\ &+\frac{1}{2}\epsilon_{abcd}\mathcal{R}^{cd}\left(\bar{\psi}\Gamma\chi - \bar{\chi}\Gamma\psi - \frac{1}{2}\mathrm{d}\left(\bar{\chi}\Gamma\chi\right) + \frac{1}{2}e_b\bar{\chi}\Gamma^b\chi\right)\\ &\left.-\epsilon_{abcd}T^c\left(\bar{\psi}\Gamma^d\chi - \bar{\chi}\Gamma^d\psi - \frac{1}{2}\mathrm{D}\left(\bar{\chi}\Gamma^d\chi\right) - \frac{1}{2}e^d\left(\bar{\chi}\Gamma\chi\right)\right)\right]\,,\end{aligned} \tag{5.13}$$

where D denotes the covariant derivative defined with respect to the four dimensional spin connection ω^{ab}.

JHEP05(2024)248

6 Non-relativistic limit

6.1 Chern–Simons theory

Let us now consider a non-relativistic contraction of the gauge algebra (3.1). We split Lorentz index A in the space-time components as $A=(0,I)$ with $I=1,\ldots,4$. Moreover, we rename and perform the following rescaling on the Poincaré superalgebra generators as in [40, 41]

$$\begin{aligned} P_0 &\longrightarrow H & P_I &\longrightarrow \lambda P_I\,, & J_{0I} &\longrightarrow \lambda G_I\,,\\ Q^\alpha &\longrightarrow \sqrt{\lambda}Q^\alpha\,, & \bar{Q}_\alpha &\longrightarrow \sqrt{\lambda}\bar{Q}_\alpha\,, & K &\longrightarrow \lambda K\,. \end{aligned} \tag{6.1}$$

When taking the limit $\lambda\to\infty$, the superalgebra (anti)commutation relations become

$$\begin{aligned} [J_{IJ},P_K] &= \eta_{JK}P_I-\eta_{IK}P_J\,,\\ [G_I,H] &= P_I\,,\\ [J_{IJ},J_{KL}] &= \eta_{JK}J_{IL}+\eta_{IL}J_{JK}-\eta_{IK}J_{JL}-\eta_{JL}J_{IK}\,,\\ [J_{IJ},G_K] &= \eta_{JK}G_I-\eta_{IK}G_J\,,\\ [J_{IJ},Q^\alpha] &= -\frac{1}{2}\left(\Gamma_{IJ}\right)^\alpha{}_\beta Q^\beta\,,\\ \left[J_{IJ},\bar{Q}_\alpha\right] &= \frac{1}{2}\left(\Gamma_{IJ}\right)^\beta{}_\alpha \bar{Q}_\beta\,,\\ \left\{Q^\alpha,\bar{Q}_\beta\right\} &= 2\left(\Gamma^I\right)^\alpha{}_\beta P_I-4i\delta^\alpha_\beta K\,. \end{aligned} \tag{6.2}$$

The non-relativistic limit of the Poincaré superalgebra (3.1) reproduces a supersymmetric extension of the Galilei algebra [42]. We now introduce a one-form gauge connection A and the corresponding gauge curvature F, to whose components we denote

$$A=\tau H+h^IP_I+\omega^IG_I+\frac{1}{2}\omega^{IJ}J_{IJ}+bK+\bar{\psi}_\alpha Q^\alpha-\bar{Q}_\alpha\psi^\alpha\,, \tag{6.3}$$

$$F=\hat{T}H+\hat{T}^IP_I+R^IG_I+\frac{1}{2}\mathcal{R}^{IJ}J_{IJ}+F_bK+\bar{\mathcal{F}}_\alpha Q^\alpha-\bar{Q}_\alpha\mathcal{F}^\alpha\,. \tag{6.4}$$

The components of the new gauge curvature are explicitly given by

$$\begin{aligned} \hat{T} &= \mathrm{d}\tau\,,\\ \hat{T}^I &= \mathrm{D}_\omega h^I+\omega^I\tau-2\bar{\psi}_\alpha\left(\Gamma^I\right)^\alpha{}_\beta\psi^\alpha\,,\\ \mathcal{R}^I &= \mathrm{D}_\omega\omega^I\,,\\ \mathcal{R}^{IJ} &= \mathrm{d}\omega^{IJ}+\omega^I{}_K\omega^{KJ}\,,\\ F_b &= \mathrm{d}b+4i\delta^\alpha_\beta\bar{\psi}_\alpha\psi^\alpha\,,\\ \bar{\mathcal{F}}_\alpha &= \mathrm{D}_\omega\bar{\psi}_\alpha,\\ \mathcal{F}^\alpha &= \mathrm{D}_\omega\psi^\alpha\,, \end{aligned} \tag{6.5}$$

where D_ω is the covariant derivative with respect to the spatial spin connection ω^{IJ}.

At the level of the invariant tensor, one can check that the non-relativistic limit of the non-vanishing components (3.2) reproduces

$$\langle J_{IJ}J_{KL}H\rangle = \frac{1}{2}\epsilon_{IJKL}\,, \tag{6.6}$$

with the convention $\epsilon_{0IJKL} = \epsilon_{IJKL}$. Then, the five-dimensional CS Lagrangian takes the form

$$\mathcal{L}^{\rm NR}_{\rm CS} = \frac{\kappa}{4}\epsilon_{IJKL}\mathcal{R}^{IJ}\mathcal{R}^{KL}\tau\,. \tag{6.7}$$

Note that, although the non-relativistic limit of the Poincaré superalgebra is a supersymmetric extension of the Galilei algebra, the CS Lagrangian does not lead to supergravity. This is a consequence of the fact that, in this limit, the invariant tensor of the algebra only carries non-zero components in the bosonic sector. As we shall see in the next section, the Carrollian limit preserves supergravity. In a future work, it would be interesting to explore the existence of a non-relativistic counterpart of Poincaré supergravity that preserves supersymmetry.

6.2 gWZW model

Let us now consider the non-relativistic limit of the gWZW action principle obtained in section 4. As we did in the relativistic case, in order to construct the transgression field theory, we introduce a secondary gauge connection A^z related with A by means of a gauge transformation. We denote the components of A^z and the supergroup parameters as follows:

$$\begin{aligned} A^z &= \tau H + H^I P_I + W^I G_I + \frac{1}{2}W^{IJ}J_{IJ} + BK + \bar{\Psi}_\alpha Q^\alpha - \bar{Q}_\alpha \Psi^\alpha\,,\\ z &= e^{-\bar{\chi}_\alpha Q^\alpha} e^{\bar{Q}_\beta \chi^\beta} e^{-\varphi K} e^{-\phi^I P_I} e^{-\phi H}\,. \end{aligned} \tag{6.8}$$

The four dimensional action is reduced to

$$\mathcal{L}^{\rm NR}_{\rm gWZW} = \kappa\epsilon_{IJKL}\mathcal{R}^{IJ}\mathcal{R}^{KL}\phi, \tag{6.9}$$

where we rename $\phi^0 = \phi$. We now perform a second index decomposition; the spatial index of the five-dimensional theory is split as $I = (i, 4)$ where $i = 1, 2, 3$ is the spatial index of the non-relativistic four-dimensional theory. We also decompose the non-relativistic spin connection and curvature as follows:

$$\begin{aligned} \omega^I &= \left(\omega^i, \tau\right)\,,\\ \omega^{IJ} &= \left(\omega^{ij}, \omega^{i4}\right) \equiv \left(\omega^{ij}, e^i\right)\,,\\ \mathcal{R}^I &= \left(\mathcal{R}^i, T\right)\,,\\ \mathcal{R}^{IJ} &= \left(\mathcal{R}^{ij}, \mathcal{R}^{i4}\right) \equiv \left(\mathcal{R}^{ij}, T^i\right)\,, \end{aligned} \tag{6.10}$$

with

$$\begin{aligned} \mathcal{R}^{ij} &= R^{ij} - e^i e^j\,,\\ T^i &= {\rm d}e^i + \omega^i{}_k e^k, \end{aligned} \tag{6.11}$$

JHEP05(2024)248

where $R^{ij} = \mathrm{d}\omega^{ij} + \omega^{i}{}_{k}\omega^{kj}$ is the $\mathfrak{so}(3)$ curvature. Since h^A is not longer present in the theory, we interpret e^i as spatial vielbein. In this way, the Lagrangian density is written as

$$\mathcal{L}^{\mathrm{NR}}_{\mathrm{gWZW}} = 4\kappa\epsilon_{ijk}\mathcal{R}^{ij}\mathcal{R}^{k4}\phi = 4\kappa\epsilon_{ijk}\left(R^{ij} - e^i e^j\right)T^k\phi\,. \tag{6.12}$$

As it happens in the non-relativistic five-dimensional CS theory, the resulting Lagrangian density is purely bosonic.

7 Ultra-relativistic limit

7.1 Chern–Simons theory

Let us now consider the ultra relativistic limit of the five-dimensional Poincaré superalgebra [26, 42, 43]. With this purpose, we consider again the space-time splitting of the Lorentz index A in the Poincaré superalgebra. We rename and rescale the generators as

$$\begin{aligned} &P_0 \longrightarrow \lambda H\,, && J_{0I} \longrightarrow \lambda G_I\,, && K \longrightarrow \lambda K\,,\\ &Q^\alpha \longrightarrow \sqrt{\lambda}Q^\alpha\,, && \bar{Q}_\alpha \longrightarrow \sqrt{\lambda}\bar{Q}_\alpha\,. \end{aligned} \tag{7.1}$$

The resulting ultra-relativistic superalgebra is obtained by taking the limit $\lambda \to \infty$, and is given by the following supersymmetric extension of the Carroll algebra [42] in five dimensions

$$\begin{aligned} [J_{IJ}, P_K] &= \eta_{JK}P_I - \eta_{IK}P_J\,,\\ [G_I, P_J] &= \eta_{IJ}H\,,\\ [J_{IJ}, J_{KL}] &= \eta_{JK}J_{IL} + \eta_{IL}J_{JK} - \eta_{IK}J_{JL} - \eta_{JL}J_{IK}\,,\\ [J_{IJ}, G_K] &= \eta_{JK}G_I - \eta_{IK}G_J\,,\\ [J_{IJ}, Q^\alpha] &= -\frac{1}{2}\left(\Gamma_{IJ}\right)^\alpha{}_\beta Q^\beta\,,\\ \left[J_{IJ}, \bar{Q}_\alpha\right] &= \frac{1}{2}\left(\Gamma_{IJ}\right)^\beta{}_\alpha \bar{Q}_\beta\,,\\ \left\{Q^\alpha, \bar{Q}_\beta\right\} &= 2\left(\Gamma^0\right)^\alpha{}_\beta H - 4i\delta^\alpha_\beta K\,. \end{aligned} \tag{7.2}$$

Following the procedure used in the non-relativistic case, we now introduce a one-form gauge connection A and the corresponding gauge curvature $F = \mathrm{d}A + A^2$. Since the ultra-relativistic algebra has the same number of generators as its non-relativistic analogue, we denote the components of A and F as they are given in eqs. (6.3) and (6.4) respectively. In this case, the components of the gauge curvature are given by

$$\begin{aligned} \hat{T} &= \mathrm{d}\tau + \omega_J h^J - 2\bar{\psi}_\alpha\left(\Gamma^0\right)^\alpha{}_\beta\psi^\alpha\,,\\ \hat{T}^I &= \mathrm{D}_\omega h^I\,,\\ \mathcal{R}^{IJ} &= \mathrm{d}\omega^{IJ} + \omega^I{}_K\omega^{KJ}\,,\\ \mathcal{R}^I &= \mathrm{D}_\omega\omega^I\,,\\ F_b &= \mathrm{d}b + 4i\delta^\alpha_\beta\bar{\psi}_\alpha\psi^\alpha\,,\\ \bar{\mathcal{F}}_\alpha &= \mathrm{D}_\omega\bar{\psi}_\alpha\,,\\ \mathcal{F}^\alpha &= \mathrm{D}_\omega\psi^\alpha\,. \end{aligned} \tag{7.3}$$

The CS Lagrangian invariant under the transformation of this algebra is given by

$$\mathcal{L}_{\rm CS}^{\rm UR} = \kappa\epsilon_{IJKL}\left(\frac{1}{4}\mathcal{R}^{IJ}\mathcal{R}^{KL}\tau + \mathcal{R}^{IJ}\mathcal{R}^{K}h^{L}\right) + \frac{i\kappa}{4}\mathcal{R}^{IJ}\mathcal{R}_{IJ}b - \kappa\left(\bar{\psi}\mathcal{R}^{IJ}\Gamma_{IJ}{\rm D}_\omega\psi + {\rm D}_\omega\bar{\psi}\mathcal{R}^{IJ}\Gamma_{IJ}\psi\right) . \tag{7.4}$$

In contrast with the non-relativistic Lagrangian, the invariant tensor still carry non-vanishing components in the fermionic sector after taking the limit and, as a consequence, the supergravity is preserved in the ultra-relativistic regime.

7.2 gWZW model

Let us now consider the ultra-relativistic limit of the four dimensional gWZW Lagrangian. As before, we introduce a group element z and a non-linear gauge field A^z, obtained from A through a large gauge transformation. We denote to the components of z and A^z as in eq. (6.8) respectively. In this case, the four-dimensional Lagrangian is reduced to

$$\mathcal{L}_{\rm gWZW}^{\rm UR} = \kappa\Big[\epsilon_{IJKL}\mathcal{R}^{IJ}\mathcal{R}^{KL}\phi + 4\epsilon_{IJKL}\mathcal{R}^{IJ}\mathcal{R}^{K}\phi^{L} + i\mathcal{R}^{IJ}\mathcal{R}_{IJ}\varphi - 8\left(\bar{\chi}\mathcal{R}^{IJ}\Gamma_{IJ}{\rm D}_\omega\psi + {\rm D}_\omega\bar{\psi}\mathcal{R}^{IJ}\Gamma_{IJ}\chi - \frac{1}{2}{\rm D}_\omega\bar{\chi}\mathcal{R}^{IJ}\Gamma_{IJ}{\rm D}_\omega\chi\right)\Big] . \tag{7.5}$$

As before, we shall decompose the five-dimensional spatial index, following the structure and change of notation of eqs. (6.10) and (6.11). In these terms, the four-dimensional ultra-relativistic gWZW Lagrangian is given by

$$\mathcal{L}_{\rm gWZW}^{\rm UR} = \kappa\Big[4\epsilon_{ijk}\mathcal{R}^{ij}T^{k}\phi + 8\epsilon_{ijk}T^{i}\mathcal{R}^{j}\phi^{k} + 4\epsilon_{ijk}\mathcal{R}^{ij}\mathcal{R}^{k}\rho - 4\epsilon_{ijk}\mathcal{R}^{ij}T\phi^{k} + i\mathcal{R}^{ij}\mathcal{R}_{ij}\varphi + 2iT^{i}T_{i}\varphi - 8\Big(\bar{\chi}\mathcal{R}^{ij}\Gamma_{ij}{\rm D}_\omega\psi + 2\bar{\chi}T^{i}\Gamma_{i}\Gamma{\rm D}_\omega\psi + {\rm D}_\omega\bar{\psi}\mathcal{R}^{ij}\Gamma_{ij}\chi + 2{\rm D}_\omega\bar{\psi}T^{i}\Gamma_{i}\Gamma\chi - \frac{1}{2}{\rm D}_\omega\bar{\chi}\mathcal{R}^{ij}\Gamma_{ij}{\rm D}_\omega\chi - {\rm D}_\omega\bar{\chi}T^{i}\Gamma_{i}\Gamma{\rm D}_\omega\chi\Big)\Big] , \tag{7.6}$$

with the convention $\epsilon_{1234} = 1$ and $\phi^4 = \rho$.

8 Concluding remarks

In this article, we have obtained a four-dimensional theory for supergravity. The construction of the action makes use of the five-dimensional $\mathcal{N} = 1$ Poincaré supergroup as gauge group, a one form gauge connection evaluated on it, and a transformed gauge connection in which the gauge parameter is evaluated in the coset space resulting between the five-dimensional Poincaré superalgebra and its Lorentz algebra. The existence of such transformed connection is enough to formulate five-dimensional standard supergravity as a gauge invariant theory of the Poincaré supergroup by means of the SW-GN formalism. This is carried out by interpreting the new connection as the fundamental field of a supergravity theory (instead of an equivalent gauge connection associated to the original one by means of a symmetry transformation), which in this case means to consider the transformed one-form V^A associated to the translation generators as fünfbein. Furthermore, we have also introduced a transgression field theory that leads to a gWZW model. As it happens in the SW-GN formalim, the field content of such model is given by the original one-form gauge connection in addition to the parameters zero-forms. The resulting action principle is fully gauge-invariant under the Poincaré supergroup and corresponds to a supersymmetric extension of the even-dimensional topological gravity introduced by A. H. Chamseddine in ref. [10].

By considering the above-mentioned gWZW supergravity theory as a starting point, we have studied two specific regimes, namely, the non- and ultra-relativistic limits. For the first one, we have found a $\mathcal{N} = 1$ supersymmetric extension of the Galilei algebra in five dimensions. Moreover, we derived a non-linear connection that allows the formulation of five-dimensional gauge invariant theories by means of the SW-GN formalism, and also obtained the corresponding four-dimensional gWZW model. We have found that, in this regime, the resulting gWZW model is not supersymmetric. Thus, we have obtained a non-relativistic gravity theory in four dimensions, which is invariant under the bosonic sector of the mentioned non-relativistic algebra. On the other hand, for the second case, we have found a five-dimensional supersymmetric extension of the Carroll algebra, obtained a non-linear realization of it, and finally derived and a four-dimensional gWZW action principle that, in contrast with the non-relativistic case, preserves supergravity.

It would be interesting to consider the semigroup expansion method [44] in order to derive a non-relativistic five-dimensional supergravity action. As it was shown in [45], some semigroups are useful to derive non-relativistic algebras with a non-degenerate invariant tensor. It would be worth exploring if such particularity can be extended in presence of supersymmetry. In particular, following the examples obtained in three spacetime dimensions [46–49], one could explore the construction of a trully supersymmetric gravity action in five spacetime dimensions along its dimensional reduction.

It would also be interesting to extend the study and the analysis done in this work to the case of a non-vanishing cosmological constant in the starting supergravity theory in five-dimensions [17]. Another aspect that deserves to be studied is the generalization of our results to $\mathcal{N}$-extended supergravities, with and without cosmological constant. Finally, it would be worth extending the analysis done along this work to the case of hypergravity, both in three and five spacetime dimensions [50–55].

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References

[1] C. Lanczos, *A remarkable property of the Riemann-Christoffel tensor in four dimensions*, *Annals Math.* **39** (1938) 842 [INSPIRE].

[2] D. Lovelock, *The Einstein tensor and its generalizations*, *J. Math. Phys.* **12** (1971) 498 [INSPIRE].

[3] B. Zumino, *Gravity Theories in More Than Four-Dimensions*, *Phys. Rept.* **137** (1986) 109 [INSPIRE].

[4] C. Teitelboim and J. Zanelli, *Dimensionally continued topological gravitation theory in Hamiltonian form*, *Class. Quant. Grav.* **4** (1987) L125 [INSPIRE].

[5] A. Mardones and J. Zanelli, *Lovelock-Cartan theory of gravity*, *Class. Quant. Grav.* **8** (1991) 1545 [INSPIRE].

[6] J. Crisostomo, R. Troncoso and J. Zanelli, *Black hole scan*, *Phys. Rev. D* **62** (2000) 084013 [hep-th/0003271] [INSPIRE].

[7] R. Troncoso and J. Zanelli, *Higher dimensional gravity, propagating torsion and AdS gauge invariance*, *Class. Quant. Grav.* **17** (2000) 4451 [hep-th/9907109] [INSPIRE].

[8] T.S. Assimos and R.F. Sobreiro, *Constrained gauge-gravity duality in three and four dimensions*, *Eur. Phys. J. C* **80** (2020) 20 [arXiv:1902.03392] [INSPIRE].

[9] M. Banados, R. Troncoso and J. Zanelli, *Higher dimensional Chern-Simons supergravity*, *Phys. Rev. D* **54** (1996) 2605 [gr-qc/9601003] [INSPIRE].

[10] A.H. Chamseddine, *Topological gravity and supergravity in various dimensions*, *Nucl. Phys. B* **346** (1990) 213 [INSPIRE].

[11] A.H. Chamseddine, *Topological Gauge Theory of Gravity in Five-dimensions and All Odd Dimensions*, *Phys. Lett. B* **233** (1989) 291 [INSPIRE].

[12] A.H. Chamseddine and D. Wyler, *Gauge Theory of Topological Gravity in* $(1+1)$*-Dimensions*, *Phys. Lett. B* **228** (1989) 75 [INSPIRE].

[13] P. Mora, *Transgression forms as unifying principle in field theory*, Other thesis, Universidad de la Republica, 11400 Montevideo, Uruguay (2003) [hep-th/0512255] [INSPIRE].

[14] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Transgression forms and extensions of Chern-Simons gauge theories*, *JHEP* **02** (2006) 067 [hep-th/0601081] [INSPIRE].

[15] F. Izaurieta, E. Rodríguez and P. Salgado, *On transgression forms and Chern-Simons (super)gravity*, hep-th/0512014 [INSPIRE].

[16] A. Borowiec, L. Fatibene, M. Ferraris and M. Francaviglia, *Covariant Lagrangian formulation of Chern-Simons and BF theories*, *Int. J. Geom. Meth. Mod. Phys.* **3** (2006) 755 [hep-th/0511060] [INSPIRE].

[17] F. Izaurieta, E. Rodríguez and P. Salgado, *The Extended Cartan homotopy formula and a subspace separation method for Chern-Simons supergravity*, *Lett. Math. Phys.* **80** (2007) 127 [hep-th/0603061] [INSPIRE].

[18] F. Izaurieta, E. Rodríguez and P. Salgado, *Eleven-dimensional gauge theory for the M algebra as an Abelian semigroup expansion of* $\mathfrak{osp}(32|1)$, *Eur. Phys. J. C* **54** (2008) 675 [hep-th/0606225] [INSPIRE].

[19] A. Anabalon, S. Willison and J. Zanelli, *General relativity from a gauged WZW term*, *Phys. Rev. D* **75** (2007) 024009 [hep-th/0610136] [INSPIRE].

JHEP05(2024)248

[20] A. Anabalon, S. Willison and J. Zanelli, *The Universe as a topological defect*, *Phys. Rev. D* **77** (2008) 044019 [hep-th/0702192] [INSPIRE].

[21] S. Salgado, F. Izaurieta, N. González and G. Rubio, *Gauged Wess-Zumino-Witten actions for generalized Poincaré algebras*, *Phys. Lett. B* **732** (2014) 255 [INSPIRE].

[22] A. Anabalon, *Some considerations on the Mac Dowell-Mansouri action*, *JHEP* **06** (2008) 069 [arXiv:0805.3558] [INSPIRE].

[23] P. Salgado, P. Salgado-Rebolledo and O. Valdivia, *Topological gravity and gauged Wess-Zumino-Witten term*, *Phys. Lett. B* **728** (2014) 99 [arXiv:1311.2532] [INSPIRE].

[24] C. Inostroza, A. Salazar and P. Salgado, *Brans–Dicke gravity theory from topological gravity*, *Phys. Lett. B* **734** (2014) 377 [INSPIRE].

[25] P. Salgado, R.J. Szabo and O. Valdivia, *Topological gravity and transgression holography*, *Phys. Rev. D* **89** (2014) 084077 [arXiv:1401.3653] [INSPIRE].

[26] C. Duval, G.W. Gibbons, P.A. Horvathy and P.M. Zhang, *Carroll versus Newton and Galilei: two dual non-Einsteinian concepts of time*, *Class. Quant. Grav.* **31** (2014) 085016 [arXiv:1402.0657] [INSPIRE].

[27] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities. Second edition (February 2008)*, in the proceedings of the *7th Mexican Workshop on Particles and Fields*, Merida, Mexico, November 10–17 (1999) [hep-th/0502193] [INSPIRE].

[28] K.S. Stelle and P.C. West, *Spontaneously Broken De Sitter Symmetry and the Gravitational Holonomy Group*, *Phys. Rev. D* **21** (1980) 1466 [INSPIRE].

[29] E.A. Ivanov and J. Niederle, *Gauge Formulation of Gravitation Theories. I. The Poincaré, De Sitter and Conformal Cases*, *Phys. Rev. D* **25** (1982) 976 [INSPIRE].

[30] E.A. Ivanov and J. Niederle, *Gauge Formulation of Gravitation Theories. II. The Special Conformal Case*, *Phys. Rev. D* **25** (1982) 988 [INSPIRE].

[31] G. Grignani and G. Nardelli, *Gravity and the Poincaré group*, *Phys. Rev. D* **45** (1992) 2719 [INSPIRE].

[32] S.R. Coleman, J. Wess and B. Zumino, *Structure of phenomenological Lagrangians. 1*, *Phys. Rev.* **177** (1969) 2239 [INSPIRE].

[33] C.G. Callan Jr., S.R. Coleman, J. Wess and B. Zumino, *Structure of phenomenological Lagrangians. II*, *Phys. Rev.* **177** (1969) 2247 [INSPIRE].

[34] A. Salam and J.A. Strathdee, *Nonlinear realizations. I: The Role of Goldstone bosons*, *Phys. Rev.* **184** (1969) 1750 [INSPIRE].

[35] P. Salgado, M. Cataldo and S. del Campo, *Supergravity and the Poincaré group*, *Phys. Rev. D* **65** (2002) 084032 [gr-qc/0110097] [INSPIRE].

[36] P. Salgado, F. Izaurieta and E. Rodríguez, *Higher dimensional gravity invariant under the AdS group*, *Phys. Lett. B* **574** (2003) 283 [hep-th/0305180] [INSPIRE].

[37] J. Manes, R. Stora and B. Zumino, *Algebraic Study of Chiral Anomalies*, *Commun. Math. Phys.* **102** (1985) 157 [INSPIRE].

[38] M. Nakahara, *Geometry, Topology and Physics*, CRC Press (2003) [DOI:10.1201/9781315275826].

[39] B. Zumino, *Nonlinear Realization of Supersymmetry in de Sitter Space*, *Nucl. Phys. B* **127** (1977) 189 [INSPIRE].

JHEP05(2024)248

[40] R. Andringa, E. Bergshoeff, S. Panda and M. de Roo, *Newtonian Gravity and the Bargmann Algebra*, *Class. Quant. Grav.* **28** (2011) 105011 [arXiv:1011.1145] [INSPIRE].

[41] N. González, G. Rubio, P. Salgado and S. Salgado, *Generalized Galilean algebras and Newtonian gravity*, *Phys. Lett. B* **755** (2016) 433 [arXiv:1604.06313] [INSPIRE].

[42] H. Bacry and J. Lévy-Leblond, *Possible kinematics*, *J. Math. Phys.* **9** (1968) 1605 [INSPIRE].

[43] J.-M. Lévy-Leblond, *Une nouvelle limite non-relativiste du groupe de poincaré*, in *Ann. Inst. H. Poincaré Phys. Theor.* **3** (1965) 1.

[44] F. Izaurieta, E. Rodríguez and P. Salgado, *Expanding Lie (super)algebras through Abelian semigroups*, *J. Math. Phys.* **47** (2006) 123512 [hep-th/0606215] [INSPIRE].

[45] P. Concha, D. Pino, L. Ravera and E. Rodríguez, *Extended kinematical 3D gravity theories*, *JHEP* **01** (2024) 040 [arXiv:2310.01335] [INSPIRE].

[46] P. Concha, L. Ravera and E. Rodríguez, *Three-dimensional non-relativistic extended supergravity with cosmological constant*, *Eur. Phys. J. C* **80** (2020) 1105 [arXiv:2008.08655] [INSPIRE].

[47] P. Concha, M. Ipinza, L. Ravera and E. Rodríguez, *Non-relativistic three-dimensional supergravity theories and semigroup expansion method*, *JHEP* **02** (2021) 094 [arXiv:2010.01216] [INSPIRE].

[48] P. Concha, L. Ravera and E. Rodríguez, *Three-dimensional exotic Newtonian supergravity theory with cosmological constant*, *Eur. Phys. J. C* **81** (2021) 646 [arXiv:2104.12908] [INSPIRE].

[49] P. Concha, L. Ravera and E. Rodríguez, *Three-dimensional non-relativistic supergravity and torsion*, *Eur. Phys. J. C* **82** (2022) 220 [arXiv:2112.05902] [INSPIRE].

[50] B. Chen, J. Long and Y.-N. Wang, *Conical Defects, Black Holes and Higher Spin (Super-)Symmetry*, *JHEP* **06** (2013) 025 [arXiv:1303.0109] [INSPIRE].

[51] Y.M. Zinoviev, *Hypergravity in AdS$_3$*, *Phys. Lett. B* **739** (2014) 106 [arXiv:1408.2912] [INSPIRE].

[52] M. Henneaux, A. Perez, D. Tempo and R. Troncoso, *Hypersymmetry bounds and three-dimensional higher-spin black holes*, *JHEP* **08** (2015) 021 [arXiv:1506.01847] [INSPIRE].

[53] O. Fuentealba, J. Matulich and R. Troncoso, *Extension of the Poincaré group with half-integer spin generators: hypergravity and beyond*, *JHEP* **09** (2015) 003 [arXiv:1505.06173] [INSPIRE].

[54] O. Fuentealba, J. Matulich and R. Troncoso, *Asymptotically flat structure of hypergravity in three spacetime dimensions*, *JHEP* **10** (2015) 009 [arXiv:1508.04663] [INSPIRE].

[55] O. Fuentealba, J. Matulich and R. Troncoso, *Hypergravity in five dimensions*, *Phys. Rev. D* **101** (2020) 124002 [arXiv:1910.03179] [INSPIRE].